An Optimal Divisioning Technique to Stabilization Synthesis of T–S Fuzzy Delayed Systems

Xiaojie Su, *Member, IEEE*, Hongying Zhou, and Yong-Duan Song, *Senior Member, IEEE*

***Abstract*—This paper investigates the problem of stability analysis and stabilization for Takagi–Sugeno (T–S) fuzzy systems with time-varying delay. By using appropriately chosen Lyapunov–Krasovskii functional, together with the reciprocally convex a new sufficient stability condition with the idea of delay partitioning approach is proposed for the delayed T–S fuzzy systems, which significantly reduces conservatism as compared with the existing results. On the basis of the obtained stability condition, the state-feedback fuzzy controller via parallel distributed compensation law is developed for the resulting fuzzy delayed systems. Furthermore, the parameters of the proposed fuzzy controller are derived in terms of linear matrix inequalities, which can be easily obtained by the optimization techniques. Finally, three examples (one of them is the benchmark inverted pendulum) are used to verify and illustrate the effectiveness of the proposed technique.**

***Index Terms*—Delay partitioning technique, fuzzy control, Takagi–Sugeno (T–S) fuzzy systems, time-varying delay.**

I. Introduction

MANY real-world systems are nonlinear and complicated, which are formidable to analyze and synthesize by utilizing the conventional theory and approach. However, with the advent of fuzzy modeling, it has been proved that complex nonlinear systems can be approximated to any accuracy by Takagi–Sugeno (T–S) fuzzy systems. The resulting fuzzy system is described by a family of IF-THEN rules to express local input–output relations of the nonlinear systems, and eventually realized by smoothly blending these local linear models together through the membership functions. Owing to the promising approximation ability of T–S fuzzy systems [12], [15], [18], [34], the research concerning fuzzy systems has appealed more and more concern and a large number of results have been reported for T–S fuzzy systems as time goes on. For example, the stability analysis and stabilization are studied in [7], [9], [10], and [29]; filtering problems are investigated in [22] and [35]; fault detection is reported in [5] and [28], and model reduction is presented in [23] and [24].

On the other hand, it is noted that the time delay, which is generally considered as one of primary sources to trigger poor performance, oscillation and even instability [11], [21], [25], frequently appears in vast engineering systems, interaction networks, chemical processes, and other areas. Inspired by this, a majority of scholars devote their efforts to analyze and synthesize the delayed T–S fuzzy systems. So far, a large number of obtained results concerning fuzzy systems with a constant time delay have been developed in [1], [4], and [8], whose application is probably limited in practical engineering since the delay in most practical situations is time-varying. Thus, it prompts us to focus on the problem of stability and stabilization of T–S fuzzy systems with time-varying delay, which are more accordant to actual systems. Existing results regarding stability analysis can generally be divided into two types. One is delay-independent approach which is irrelevant to the size of delay when analyzing and synthesizing systems. The other is delay-dependent approach which cares much about the information of time delay, but this one is less conservative than the former especially when the delay size is small. Thereby, it is extremely worth investigating the nonlinear systems using the T–S fuzzy models with the delay-dependent technique.

With the recent progress of advanced mathematical and robust control techniques, a multitude of methods have been proposed to develop delay-dependent conditions for the time-delay systems. For example, the approach is to divide the time delay into multiple segments uniformly for the constant time-delay T–S fuzzy systems proposed in [31]. The work in [32] presents a delay partitioning technique which partitions the time delay range $[0, d(t)]$ for the linear delayed systems. The approach used in [3] divides the whole delay interval into l subintervals, which is proved to be less and less conservative as the partitioning segments become increasingly thinner for the concerned systems. Also, the input–output technique, borrowed from the robust control theory in [22], is one of the most available ways to deal with the time delay. However, there are quiet a few results for T–S fuzzy time-varying delay systems when the time delay becomes uncertain. Thus, how to further improve current results affected by time delay and reduce the complexity in the calculation employing the above methods is still a paramount and challenging problem.

Manuscript received October 27, 2015; revised January 8, 2016; accepted February 22, 2016. Date of publication April 5, 2016; date of current version April 13, 2017. This work was supported in part by the National Key Basic Research Program (973), China, under Grant 2012CB215202 and Grant 2014CB249200, in part by the National Natural Science Foundation of China under Grant 61403048 and Grant 61134001, in part by the Basic and Frontier Research Project of Chongqing under Grant cstc2015jcyjA40005, and in part by the Fundamental Research Funds for the Central Universities under Grant 106112015CDJXY170001. This paper was recommended by Associate Editor C. W. Tao. *(Corresponding author: Yong-Duan Song.)*

The authors are with the Key Laboratory of Dependable Service Computing in Cyber Physical Society of Ministry of Education, Chongqing University, Chongqing 400044, China, and also with the College of Automation, Chongqing University, Chongqing 400044, China (e-mail: suxiaojie@cqu.edu.cn; hyzhou@cqu.edu.cn; ydsong@cqu.edu.cn).

Digital Object Identifier 10.1109/TCYB.2016.2538464

Based on the above discussions, the aim of this paper is to develop a new delay-dependent condition to solve the problem of stability analysis and stabilization for T–S fuzzy systems with time-varying delay. First, the sufficient condition of stability analysis for the fuzzy systems with time-varying delay is derived by the reciprocally convex approach to amplify the convex combination of quadratic terms resulted from the integral terms of quadratic quantities. Then, based on the parallel distributed compensation (PDC) scheme in [2], [6], and [33], the state-feedback fuzzy controller for the derived fuzzy systems is designed, whose parameters can be easily obtained by the linear matrix inequality (LMI) optimization technique. Finally, three simulation examples are given to demonstrate the effectiveness and advantages of the proposed approaches in this paper. The main contributions of this paper, which principally owing to the usage of the delay partitioning technique, are summarized as follows.

1) To reduce the conservatism of the proposed stability analysis method, the entire delay interval is divided into multiple segments and each delay component is taken into account during the construction of Lyapunov–Krasovskii functional, furthermore, the effectiveness of this partitioning approach is shown via the detailed example.
2) To further tackle the conservatism generated by time-varying delay, the reciprocally convex technique is employed to amplify the results of the integral terms of quadratic quantities, instead of immediately ignoring the negative integral terms of quadratic quantities.
3) Some free fuzzy weighting matrices are introduced to deal with the derivation of Lyapunov–Krasovskii functional, which relax the constraints of the stability analysis effectively.

The rest of this paper are organized as follows. The system model and some preliminaries are described in Section II. In Section III, the problem of stability analysis and stabilization for T–S fuzzy systems with time-varying delay is developed. Section IV presents three examples to illustrate the effectiveness and merits of the proposed approach. Our conclusions are given in Section V.

A. Notation

The superscript "T" stands for matrix transposition; $\mathbb{R}^n$ indicates the n dimensional Euclidean space; $P > 0 (\geq 0)$ means that P is real symmetric and positive definite(semidefinite); I and 0 represents the identity matrix and zero matrix with compatible dimensions, respectively; sym(A) is defined as $A+A^{\mathrm{T}}$; diag$(\ldots)$ stands for a block-diagonal matrix; $\star$ denotes the symmetric terms in a symmetric matrix; Matrices, if their dimensions are not precisely stated, are assumed to be compatible for algebraic operations.

II. Preliminaries

In this paper, we consider a class of continuous-time nonlinear systems which can be represented by the following T–S fuzzy time-varying delay model.

A. Plant Form

Rule i: IF $\theta_1(t)$ is μ_{i1} and $\theta_2(t)$ is μ_{i2} and $\cdots$ and $\theta_p(t)$ is μ_{ip}, THEN

$$\begin{cases} \dot{x}(t) = A_i x(t) + A_{di} x(t - d(t)) + B_i u(t) \\ x(t) = \varphi(t), \quad t \in \left[-\bar{d}, 0\right], \quad i = 1, 2, \ldots, r \end{cases}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^q$ is the control input vector; $d(t)$ is a time-varying delay satisfying $0 \leq \underline{d} \leq d(t) \leq \bar{d} < \infty$, $\dot{d}(t) \leq \tau$, where $\underline{d}$, $\bar{d}$, and τ are real constant scalars. We partition the delay interval into m fractions, $[\underline{d}, \bar{d}] = \bigcup_{i=1}^{m} [d_{i-1}, d_i]$, with $d_0 = \underline{d}$, $d_m = \bar{d}$. Let η_i indicate the length of the subinterval, $\eta_i = d_i - d_{i-1}$ $i = 1, \ldots, m$, with $d_{-1} = 0$. A_i, A_{di}, and B_i are system matrices with compatible dimensions; $\theta_1(t), \theta_2(t), \ldots, \theta_p(t)$ are the premise variables; $\mu_{i1}, \ldots, \mu_{ip}$ are the fuzzy sets; r is the number of IF-THEN rules; $\varphi(t)$ denotes the initial condition.

Given a pair of $(x(t), u(t))$, the final representation of the above T–S fuzzy time-varying delay system is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(\theta(t))[A_i x(t) + A_{di} x(t - d(t)) + B_i u(t)]$$

with

$$h_i(\theta(t)) = \frac{\nu_i(\theta(t))}{\sum_{i=1}^{r} \nu_i(\theta(t))}, \nu_i(\theta(t)) = \prod_{j=1}^{p} \mu_{ij}\big(\theta_j(t)\big)$$

where $\mu_{ij}(\theta_j(t))$ represents the grade of membership of $\theta_j(t)$ in μ_{ij}. Therefore, for all t, we can see $\nu_i(\theta(t)) \geq 0$, $i = 1, 2, \ldots, r$, $\sum_{i=1}^{r} h_i(\theta(t)) = 1$. For notational simplicity, define

$$A(t) \triangleq \sum_{i=1}^{r} h_i(\theta(t)) A_i, \quad A_d(t) \triangleq \sum_{i=1}^{r} h_i(\theta(t)) A_{di}$$
$$B(t) \triangleq \sum_{i=1}^{r} h_i(\theta(t)) B_i.$$

Assume that the premise variable of the fuzzy model $\theta(t)$ is available for feedback, which implies that $h_i(\theta(t))$ is available for feedback control. Suppose that the controller's premise variables are the same as those in the plant. The PDC strategy is utilized and the fuzzy state feedback controller obeys the following rules.

B. Fuzzy Controller

Rule i: IF $\theta_1(t)$ is μ_{i1} and $\theta_2(t)$ is μ_{i2} and $\cdots$ and $\theta_p(t)$ is μ_{ip}, THEN

$$u(t) = K_i x(t), \qquad i = 1, 2, \ldots, r$$

where K_i is the gain matrix of the state-feedback controller.

The controller can be expressed as the following form:

$$u(t) = \sum_{i=1}^{r} h_i(\theta(t)) K_i x(t). \tag{1}$$

The compact form of the controller is denoted as

$$u(t) = K(t) x(t) \tag{2}$$

where $K(t) = \sum_{i=1}^{r} h_i(\theta(t))K_i$.

Thus, with the controller as defined in (1), the closed-loop system can be presented as

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\theta(t))h_j(\theta(t))[A_i x(t) + A_{di}x(t-d(t)) + B_i K_j x(t)] \quad (3)$$

or expressed in compact form

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t) + A_d(t)x(t-d(t)). \quad (4)$$

To present the main results, we introduce the following lemma.

Lemma 1 [16]: Let $f_1, f_2, \ldots, f_N : \mathbb{R}^m \to R$ have positive values in an open subset D of $\mathbb{R}^m$. Then, the reciprocally convex combination of f_i over D satisfies

$$\min_{\{\beta_i|\beta_i>0,\sum_i \beta_i=1\}} \sum_i \frac{1}{\beta_i} f_i(t) = \sum_i f_i(t) + \max_{g_{i,j}(t)} \sum_{i\neq j} g_{i,j}(t)$$

subject to

$$\left\{ g_{i,j} : \mathbb{R}^m \to \mathbb{R}, g_{j,i}(t) = g_{i,j}(t), \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{j,i}(t) & f_j(t) \end{bmatrix} \geq 0 \right\}.$$

Remark 1: In the following derivation of the Lyapunov–Krasovskii functional, the crucial issue is how to deal with the amplification obtained from the negative integral terms of quadratic quantities. Lemma 1 is evidently an efficient method to handle this terms of the Lyapunov–Krasovskii functional for the concerned delayed systems. This processing scheme was used for the time-varying delay systems in [3] and [16], which will be used in our later derivation.

Lemma 2 [20]: For any constant matrix $M > 0$, scalars $b > a > 0$, vector function $w : [a, b] \to \mathbb{R}^n$, then

$$\begin{aligned} &-(b-a)\int_{t-b}^{t-a} w^{\mathrm{T}}(s)Mw(s)ds \\ &\leq -\left(\int_{t-b}^{t-a} w(s)ds\right)^{\mathrm{T}} M \left(\int_{t-b}^{t-a} w(s)ds\right) \\ &-\frac{b^2-a^2}{2}\int_{-b}^{-a}\int_{t+\theta}^{t} w^{\mathrm{T}}(s)Mw(s)dsd\theta \\ &\leq -\left(\int_{-b}^{-a}\int_{t+\theta}^{t} w(s)dsd\theta\right)^{\mathrm{T}} M \left(\int_{-b}^{-a}\int_{t+\theta}^{t} w(s)dsd\theta\right). \end{aligned}$$

III. Main Results

A. Stability Analysis

In order to describe our method clearly, in this section, we partition the delay interval into two parts. By utilizing the reciprocally convex approach to deal with the Lyapunov functional, we establish a new stability criterion for T–S fuzzy systems with time-varying delay.

Theorem 1: Suppose the controller gain matrices in (2) are known in advance. Given scalars d_0, d_2, and τ, the delayed fuzzy system (4) is asymptotically stable, if there exist matrices $S_N > 0$, $R_N > 0$, $Q_N > 0$, $N = 0, 1, 2$, $P > 0$, $Y > 0$, $M(t)$, and Z_1 and Z_2 satisfying the following inequalities:

$$\Phi(t) + \Phi_1(t) < 0 \quad (5)$$

$$\Phi(t) + \Phi_2(t) < 0 \quad (6)$$

$$\begin{bmatrix} S_1 & Z_1 \\ \star & S_1 \end{bmatrix} \geq 0 \quad (7)$$

$$\begin{bmatrix} S_2 & Z_2 \\ \star & S_2 \end{bmatrix} \geq 0 \quad (8)$$

where

$$\begin{aligned} \Phi(t) \triangleq\ & d_0^2 W_9^{\mathrm{T}} S_0 W_9 - (W_1 - W_2)^{\mathrm{T}} S_0 (W_1 - W_2) \\ &+ \eta_1^2 W_9^{\mathrm{T}} S_1 W_9 + \eta_2^2 W_9^{\mathrm{T}} S_2 W_9 + \frac{d_0^4}{4} W_9^{\mathrm{T}} R_0 W_9 \\ &- (\eta_0 W_1 - W_6)^{\mathrm{T}} R_0 (\eta_0 W_1 - W_6) + W_1^{\mathrm{T}} Q_0 W_1 \\ &+ \frac{(d_1^2 - d_0^2)^2}{4} W_9^{\mathrm{T}} R_1 W_9 + \frac{(d_2^2 - d_1^2)^2}{4} W_9^{\mathrm{T}} R_2 W_9 \\ &- (\eta_1 W_1 - W_7)^{\mathrm{T}} R_1 (\eta_1 W_1 - W_7) - W_2^{\mathrm{T}} Q_0 W_2 \\ &+ W_1^{\mathrm{T}} Y W_1 + (\tau - 1) W_5^{\mathrm{T}} Y W_5 + \mathrm{sym}(W_1^{\mathrm{T}} P W_9) \\ &- (\eta_2 W_1 - W_8)^{\mathrm{T}} R_2 (\eta_2 W_1 - W_8) + \mathrm{sym}(M(t) W_A(t)) \\ &- W_3^{\mathrm{T}} Q_1 W_3 + W_3^{\mathrm{T}} Q_2 W_3 - W_4^{\mathrm{T}} Q_2 W_4 + W_2^{\mathrm{T}} Q_1 W_2 \end{aligned}$$

$$\begin{aligned} \Phi_1(t) \triangleq\ & -(W_5 - W_2)^{\mathrm{T}} S_1 (W_5 - W_2) \\ &- (W_5 - W_3)^{\mathrm{T}} Z_1^{\mathrm{T}} (W_5 - W_2) \\ &- (W_5 - W_2)^{\mathrm{T}} Z_1 (W_5 - W_3) \\ &- (W_5 - W_3)^{\mathrm{T}} S_1 (W_5 - W_3) \\ &- (W_3 - W_4)^{\mathrm{T}} S_2 (W_3 - W_4) \end{aligned}$$

$$\begin{aligned} \Phi_2(t) \triangleq\ & -(W_5 - W_3)^{\mathrm{T}} S_2 (W_5 - W_3) \\ &- (W_5 - W_4)^{\mathrm{T}} Z_2^{\mathrm{T}} (W_5 - W_3) \\ &- (W_5 - W_3)^{\mathrm{T}} Z_2 (W_5 - W_4) \\ &- (W_5 - W_4)^{\mathrm{T}} S_2 (W_5 - W_4) \\ &- (W_2 - W_3)^{\mathrm{T}} S_1 (W_2 - W_3) \end{aligned}$$

$$\begin{aligned} &W_1 \triangleq [I_n \quad 0_{n,8n}], && W_2 \triangleq [0_{n,n} \quad I_n \quad 0_{n,7n}] \\ &W_3 \triangleq [0_{n,2n} \quad I_n \quad 0_{n,6n}], && W_4 \triangleq [0_{n,3n} \quad I_n \quad 0_{n,5n}] \\ &W_5 \triangleq [0_{n,4n} \quad I_n \quad 0_{n,4n}], && W_6 \triangleq [0_{n,5n} \quad I_n \quad 0_{n,3n}] \\ &W_7 \triangleq [0_{n,6n} \quad I_n \quad 0_{n,2n}], && W_8 \triangleq [0_{n,7n} \quad I_n \quad 0_{n,n}] \\ &W_9 \triangleq [0_{n,8n} \quad I_n] \end{aligned}$$

$$W_A(t) \triangleq [A(t) + B(t)K(t) \quad 0_{n,3n} \quad A_d(t) \quad 0_{n,3n} \quad -I_n].$$

Proof: First, we construct the following Lyapunov–Krasovskii functional:

$$V(t) \triangleq \sum_{i=1}^{5} V_i(t)$$

where

$$\begin{aligned} V_1(t) &\triangleq x^{\mathrm{T}}(t) P x(t) \\ V_2(t) &\triangleq \sum_{i=0}^{2} \int_{t-d_i}^{t-d_{i-1}} x^{\mathrm{T}}(s) Q_i x(s) ds \\ V_3(t) &\triangleq \int_{t-d(t)}^{t} x^{\mathrm{T}}(s) Y x(s) ds + d_0 \int_{-d_0}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) S_0 \dot{x}(s) ds d\theta \\ V_4(t) &\triangleq \sum_{i=1}^{2} \eta_i \int_{-d_i}^{-d_{i-1}}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) S_i \dot{x}(s) ds d\theta \\ V_5(t) &\triangleq \sum_{i=0}^{2} \frac{d_i^2 - d_{i-1}^2}{2} \int_{-d_i}^{-d_{i-1}}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{x}^{\mathrm{T}}(s) R_i \dot{x}(s) ds d\lambda d\theta. \end{aligned}$$

The time derivative of the Lyapunov–Krasovskii functional is given by

$$\begin{aligned}
\dot{V}_1(t) &= x^{\mathrm{T}}(t)P\dot{x}(t) + \dot{x}^{\mathrm{T}}(t)Px(t)\\
\dot{V}_2(t) &= \sum_{i=0}^{2}\big\{x^{\mathrm{T}}(t-d_{i-1})Q_i x(t-d_{i-1})\\
&\qquad - x^{\mathrm{T}}(t-d_i)Q_i x(t-d_i)\big\}\\
\dot{V}_3(t) &= d_0^2\dot{x}^{\mathrm{T}}(t)S_0\dot{x}(t) - d_0\int_{t-d_0}^{t}\dot{x}^{\mathrm{T}}(s)S_0\dot{x}(s)ds + x^{\mathrm{T}}(t)Yx(t)\\
&\quad - \big(1-\dot{d}(t)\big)x^{\mathrm{T}}(t-d(t))Yx(t-d(t))\\
\dot{V}_4(t) &= \sum_{i=1}^{2}\left\{\eta_i^2\dot{x}^{\mathrm{T}}(t)S_i\dot{x}(t) - \eta_i\int_{t-d_i}^{t-d_{i-1}}\dot{x}^{\mathrm{T}}(s)S_i\dot{x}(s)ds\right\}\\
\dot{V}_5(t) &= -\sum_{i=0}^{2}\frac{d_i^2-d_{i-1}^2}{2}\int_{-d_i}^{-d_{i-1}}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_i\dot{x}(s)dsd\theta\\
&\quad + \sum_{i=0}^{2}\frac{(d_i^2-d_{i-1}^2)^2}{4}\dot{x}^{\mathrm{T}}(t)R_i\dot{x}(t).
\end{aligned}\tag{9}$$

According to Lemma 2, we get

$$\begin{aligned}
\dot{V}_3(t) &\le d_0^2\dot{x}^{\mathrm{T}}(t)S_0\dot{x}(t) - (x(t)-x(t-d_0))^{\mathrm{T}}S_0(x(t)\\
&\quad - x(t-d_0)) + x^{\mathrm{T}}(t)Yx(t)\\
&\quad - (1-\tau)x^{\mathrm{T}}(t-d(t))Yx(t-d(t))\\
\dot{V}_5(t) &\le \sum_{i=0}^{2}\frac{\left(d_i^2-d_{i-1}^2\right)^2}{4}\dot{x}^{\mathrm{T}}(t)R_i\dot{x}(t)\\
&\quad -\sum_{i=0}^{2}\left(\eta_i x(t) - \int_{t-d_i}^{t-d_{i-1}}x(s)ds\right)^{\mathrm{T}}\\
&\qquad R_i\left(\eta_i x(t) - \int_{t-d_i}^{t-d_{i-1}}x(s)ds\right).
\end{aligned}\tag{10}$$

For $\dot{V}_4(t)$, if $d_0 < d(t) < d_1$, by Lemmas 1 and 2, it can be shown that

$$\begin{aligned}
\dot{V}_4(t) &= \eta_1^2\dot{x}^{\mathrm{T}}(t)S_1\dot{x}(t) - \eta_1\int_{t-d(t)}^{t-d_0}\dot{x}^{\mathrm{T}}(s)S_1\dot{x}(s)ds\\
&\quad + \eta_2^2\dot{x}^{\mathrm{T}}(t)S_2\dot{x}(t) - \eta_1\int_{t-d_1}^{t-d(t)}\dot{x}^{\mathrm{T}}(s)S_1\dot{x}(s)ds\\
&\quad - \eta_2\int_{t-d_2}^{t-d_1}\dot{x}^{\mathrm{T}}(s)S_2\dot{x}(s)ds,\\
&\le \eta_1^2\dot{x}^{\mathrm{T}}(t)S_1\dot{x}(t) + \eta_2^2\dot{x}^{\mathrm{T}}(t)S_2\dot{x}(t)\\
&\quad - (x(t-d_1)-x(t-d_2))^{\mathrm{T}}S_2(x(t-d_1)-x(t-d_2))\\
&\quad - \frac{\eta_1}{d(t)-d_0}(x(t-d(t))-x(t-d_0))^{\mathrm{T}}S_1(x(t-d(t))\\
&\quad - x(t-d_0)) - \frac{\eta_1}{d_1-d(t)}(x(t-d(t))\\
&\quad - x(t-d_1))^{\mathrm{T}}S_1(x(t-d(t))-x(t-d_1))\\
&\le \eta_1^2\dot{x}^{\mathrm{T}}(t)S_1\dot{x}(t) + \eta_2^2\dot{x}^{\mathrm{T}}(t)S_2\dot{x}(t)\\
&\quad - (x(t-d_1)-x(t-d_2))^{\mathrm{T}}S_2(x(t-d_1)-x(t-d_2))\\
&\quad - \begin{bmatrix} x(t-d(t))-x(t-d_0)\\ x(t-d(t))-x(t-d_1)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} S_1 & Z_1\\ \star & S_1\end{bmatrix}\\
&\quad \begin{bmatrix} x(t-d(t))-x(t-d_0)\\ x(t-d(t))-x(t-d_1)\end{bmatrix}.
\end{aligned}\tag{11}$$

When $d(t) = d_0$ or $d(t) = d_1$, (11) still holds due to $x(t-d(t)) - x(t-d_0) = 0$ or $x(t-d(t)) - x(t-d_1) = 0$. On the basis of the T–S fuzzy systems in (4), we have

$$\begin{aligned}
\Delta = 2\zeta^{\mathrm{T}}(t)M(t)[(A(t)+B(t)K(t))x(t) + A_d(t)x(t-d(t))\\
- \dot{x}(t)] = 0
\end{aligned}$$

where $M(t) = \sum_{i=1}^{r}h_i(\theta(t))M_i$.

Hence, considering (9)–(11), when $\zeta(t) \ne 0$, it is not difficult to get

$$\dot{V}(t) = \dot{V}(t) + \Delta \le \zeta^{\mathrm{T}}(t)(\Phi(t)+\Phi_1(t))\zeta(t) < 0$$

where

$$\begin{aligned}
\zeta(t) \triangleq \Big[&x^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t-d_0)\ x^{\mathrm{T}}(t-d_1)\ x^{\mathrm{T}}(t-d_2)\ x^{\mathrm{T}}(t-d(t))\\
&\int_{t-d_0}^{t}x^{\mathrm{T}}(s)ds\ \int_{t-d_1}^{t-d_0}x^{T}(s)ds\ \int_{t-d_2}^{t-d_1}x^{T}(s)ds\ \dot{x}^{T}(t)\Big]^{\mathrm{T}}.
\end{aligned}$$

For $\dot{V}_4(t)$, if $d_1 < d(t) < d_2$, following the same method as above, it is readily obtained that:

$$\begin{aligned}
\dot{V}_4(t) &= \eta_1^2\dot{x}^{\mathrm{T}}(t)S_1\dot{x}(t) - \eta_1\int_{t-d_1}^{t-d_0}\dot{x}^{\mathrm{T}}(s)S_1\dot{x}(s)ds\\
&\quad + \eta_2^2\dot{x}^{\mathrm{T}}(t)S_2\dot{x}(t) - \eta_2\int_{t-d(t)}^{t-d_1}\dot{x}^{\mathrm{T}}(s)S_2\dot{x}(s)ds\\
&\quad - \eta_2\int_{t-d_2}^{t-d(t)}\dot{x}^{\mathrm{T}}(s)S_2\dot{x}(s)ds\\
&\le \eta_1^2\dot{x}^{\mathrm{T}}(t)S_1\dot{x}(t) + \eta_2^2\dot{x}^{\mathrm{T}}(t)S_2\dot{x}(t)\\
&\quad - (x(t-d_0)-x(t-d_1))^{\mathrm{T}}S_1(x(t-d_0)-x(t-d_1))\\
&\quad - \begin{bmatrix} x(t-d(t))-x(t-d_1)\\ x(t-d(t))-x(t-d_2)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix} S_2 & Z_2\\ \star & S_2\end{bmatrix}\\
&\quad \begin{bmatrix} x(t-d(t))-x(t-d_1)\\ x(t-d(t))-x(t-d_2)\end{bmatrix}.
\end{aligned}\tag{12}$$

When $d(t) = d_1$ or $d(t) = d_2$, (12) still holds due to $x(t-d(t))-x(t-d_1) = 0$ or $x(t-d(t))-x(t-d_2) = 0$. Similarly, if $d_1 < d(t) < d_2$, considering (9), (10), and (12), when $\zeta(t) \ne 0$, we obtain

$$\dot{V}(t) = \dot{V}(t) + \Delta \le \zeta^{\mathrm{T}}(t)(\Phi(t)+\Phi_2(t))\zeta(t) < 0.$$

Thus, it is proven that the fuzzy system (4) is asymptotically stable. ■

It is noted that Theorem 1 cannot be directly solved by LMI for the stability analysis. Our next aim is to convert the inequalities (5) and (6) to some finite LMIs, which can be easily solved by the standard numerical software. Therefore, we have the following theorem.

Theorem 2: Suppose the controller gain matrices K_i in (1) are known in advance. Given scalars d_0, d_2, and τ, the delayed fuzzy system (3) is asymptotically stable, if there exist matrices $S_N > 0$, $R_N > 0$, $Q_N > 0$, $N = 0, 1, 2$, $P > 0$, $Y > 0$, M_i, Z_1, and Z_2 satisfying (7) and (8) and the following inequalities for $i, j, k = 1, 2, \ldots, r$:

$$\Phi_{iik} + \Phi_{1iik} < 0 \tag{13}$$

$$\Phi_{ijk} + \Phi_{1ijk} + \Phi_{jik} + \Phi_{1jik} < 0, \quad 1 \le i < j \le r \tag{14}$$

$$\Phi_{iik} + \Phi_{2iik} < 0 \tag{15}$$

$$\Phi_{ijk} + \Phi_{2ijk} + \Phi_{jik} + \Phi_{2jik} < 0, \quad 1 \le i < j \le r \tag{16}$$

where

$$\begin{aligned}\Phi_{ijk} &\triangleq d_0^2 W_9^{\mathrm{T}} S_0 W_9 - (W_1 - W_2)^{\mathrm{T}} S_0 (W_1 - W_2)\\ &\quad + \eta_1^2 W_9^{\mathrm{T}} S_1 W_9 + \eta_2^2 W_9^{\mathrm{T}} S_2 W_9 + \frac{d_0^4}{4} W_9^{\mathrm{T}} R_0 W_9\\ &\quad - (\eta_0 W_1 - W_6)^{\mathrm{T}} R_0 (\eta_0 W_1 - W_6)\\ &\quad + \frac{(d_1^2 - d_0^2)^2}{4} W_9^{\mathrm{T}} R_1 W_9 - (\eta_1 W_1 - W_7)^{\mathrm{T}} R_1 (\eta_1 W_1 - W_7)\\ &\quad + \frac{(d_2^2 - d_1^2)^2}{4} W_9^{\mathrm{T}} R_2 W_9 + \mathrm{sym}(W_1^{\mathrm{T}} P W_9)\\ &\quad - (\eta_2 W_1 - W_8)^{\mathrm{T}} R_2 (\eta_2 W_1 - W_8)\\ &\quad + W_1^{\mathrm{T}} Y W_1 + (\tau - 1) W_5^{\mathrm{T}} Y W_5 + W_1^{\mathrm{T}} Q_0 W_1 - W_2^{\mathrm{T}} Q_0 W_2\\ &\quad + W_2^{\mathrm{T}} Q_1 W_2 - W_3^{\mathrm{T}} Q_1 W_3 + W_3^{\mathrm{T}} Q_2 W_3 - W_4^{\mathrm{T}} Q_2 W_4\\ &\quad + \mathrm{sym}(M_i W_{Ajk})\\ \Phi_{1ijk} &\triangleq -(W_5 - W_2)^{\mathrm{T}} S_1 (W_5 - W_2) - (W_5 - W_3)^{\mathrm{T}} Z_1^T (W_5 - W_2)\\ &\quad - (W_5 - W_2)^{\mathrm{T}} Z_1 (W_5 - W_3) - (W_5 - W_3)^{\mathrm{T}} S_1 (W_5 - W_3)\\ &\quad - (W_3 - W_4)^{\mathrm{T}} S_2 (W_3 - W_4)\\ \Phi_{2ijk} &\triangleq -(W_5 - W_3)^{\mathrm{T}} S_2 (W_5 - W_3) - (W_5 - W_4)^{\mathrm{T}} Z_2^T (W_5 - W_3)\\ &\quad - (W_5 - W_3)^{\mathrm{T}} Z_2 (W_5 - W_4) - (W_5 - W_4)^{\mathrm{T}} S_2 (W_5 - W_4)\\ &\quad - (W_2 - W_3)^{\mathrm{T}} S_1 (W_2 - W_3)\\ W_{Ajk} &\triangleq \left[A_j + B_j K_k \;\; 0_{n,3n} \;\; A_{dj} \;\; 0_{n,3n} \;\; -I_n\right].\end{aligned}$$

Proof: Based on the fuzzy basis functions, the inequalities (5) and (6) can be rewritten as

$$\sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t)) \sum_{k=1}^{r} h_k(\theta(t)) \big(\Phi_{ijk} + \Phi_{1ijk}\big) < 0, \quad (17)$$

$$\sum_{i=1}^{r} h_i(\theta(t)) \sum_{j=1}^{r} h_j(\theta(t)) \sum_{k=1}^{r} h_k(\theta(t)) \big(\Phi_{ijk} + \Phi_{2ijk}\big) < 0. \quad (18)$$

Therefore, if the inequalities (13)–(16) are satisfied, obviously, the aforementioned inequalities (17) and (18) hold. That is to say, the T–S fuzzy time-delay system (3) is asymptotically stable. ■

In order to further reduce the conservatism, we now extend the result to the instance whose delay interval is partitioned into arbitrarily m parts with the following Lyapunov–Krasovskii functional:

$$V(t) \triangleq \sum_{i=1}^{5} V_i(t) \quad (19)$$

where

$$\begin{aligned}V_1(t) &\triangleq x^{\mathrm{T}}(t) P x(t)\\ V_2(t) &\triangleq \sum_{i=0}^{m} \int_{t-d_i}^{t-d_{i-1}} x^{\mathrm{T}}(s) Q_i x(s) ds\\ V_3(t) &\triangleq \int_{t-d(t)}^{t} x^{\mathrm{T}}(s) Y x(s) ds + d_0 \int_{-d_0}^{0} \int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) S_0 \dot{x}(s) ds d\theta\\ V_4(t) &\triangleq \sum_{i=1}^{m} \eta_i \int_{-d_i}^{-d_{i-1}} \int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) S_i \dot{x}(s) ds d\theta\\ V_5(t) &\triangleq \sum_{i=0}^{m} \frac{d_i^2 - d_{i-1}^2}{2} \int_{-d_i}^{-d_{i-1}} \int_{\theta}^{0} \int_{t+\lambda}^{t} \dot{x}^{\mathrm{T}}(s) R_i \dot{x}(s) ds d\lambda d\theta.\end{aligned}$$

According to the aforementioned Lyapunov–Krasovskii functional, we can easily obtain the following result whose delay interval is partitioned arbitrarily m parts, which can be proved by using similar arguments as those in the proof of Theorems 1 and 2.

Theorem 3: Suppose the controller gain matrices K_i in (1) are known in advance. Given scalars d_0, d_m, $m \geq 2$, and τ, the delayed fuzzy system (3) is asymptotically stable, if there exist matrices $S_N > 0$, $R_N > 0$, $Q_N > 0$, $N = 0, 1, \ldots, m$, Z_l, $l = 1, \ldots, m$, $P > 0$, $Y > 0$, and M_i satisfying the following inequalities for $i, j, k = 1, 2, \ldots, r$:

$$\Phi_{iik} + \Phi_{liik} < 0 \quad (20)$$

$$\Phi_{ijk} + \Phi_{lijk} + \Phi_{jik} + \Phi_{ljik} < 0, \quad 1 \leq i < j \leq r \quad (21)$$

$$\begin{bmatrix} S_l & Z_l \\ \star & S_l \end{bmatrix} \geq 0, \quad l = 1, \ldots, m \quad (22)$$

where

$$\begin{aligned}\Phi_{ijk} &\triangleq \sum_{N=0}^{m} \frac{(d_N^2 - d_{N-1}^2)^2}{4} W_{2m+5}^{\mathrm{T}} R_N W_{2m+5}\\ &\quad - \sum_{N=0}^{m} (\eta_N W_1 - W_{m+4+N})^{\mathrm{T}} R_N (\eta_N W_1 - W_{m+4+N})\\ &\quad + \sum_{N=0}^{m} \big(W_{N+1}^{\mathrm{T}} Q_N W_{N+1} - W_{N+2}^{\mathrm{T}} Q_N W_{N+2}\big) + W_1^{\mathrm{T}} Y W_1\\ &\quad + \sum_{N=0}^{m} \eta_N^2 W_{2m+5}^{\mathrm{T}} S_N W_{2m+5} + \mathrm{sym}(W_1^{\mathrm{T}} P W_{2m+5})\\ &\quad - (W_1 - W_2)^{\mathrm{T}} S_0 (W_1 - W_2) + (\tau - 1) W_{m+3}^{\mathrm{T}} Y W_{m+3}\\ &\quad + \mathrm{sym}\big(M_i W_{Ajk}\big)\\ \Phi_{lijk} &\triangleq -\begin{bmatrix} W_{m+3} - W_{l+1} \\ W_{m+3} - W_{l+2} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} S_l & Z_l \\ \star & S_l \end{bmatrix} \begin{bmatrix} W_{m+3} - W_{l+1} \\ W_{m+3} - W_{l+2} \end{bmatrix}\\ &\quad - \sum_{h=1,\; h\neq l}^{m} (W_{h+1} - W_{h+2})^{\mathrm{T}} S_h (W_{h+1} - W_{h+2})\\ W_{Ajk} &\triangleq \left[A_j + B_j K_k \quad 0_{n,(m+1)n} \quad A_{dj} \quad 0_{n,(m+1)n} \quad -I_n\right]\\ W_t &\triangleq \left[0_{n,(t-1)n} \quad I_n \quad 0_{n,(2m+5-t)n}\right], t = 1, 2, \ldots, 2m+5.\end{aligned}$$

Remark 2: The proposed stability conditions do not require the limited upper bound for the delay derivative, which are more general than the results with rigid constraints $\tau < 1$. Obviously, our approach is more suitable for real systems.

Remark 3: In this paper, the algorithm complexity is closely dependent on the number of delay partitions. When the delay is partitioned into m subintervals, the decision variable number is $4m + 5 + r$. From the Examples 1–3, as the partitioning getting thinner, the conservatism reduction of Theorem 3 is more obvious. However, the merit achieved is at the sacrifice of increasing computational complexities.

Remark 4: In Theorem 3, we applied the reciprocally convex technique combining with the idea of delay partitioning approach and a novel Lyapunov functional (19) to analyze the stability of the T–S fuzzy delayed system. The main attention is focused on reduction of the conservativeness such that the controller design synthesis problems have feasible solutions. To show the effective of the proposed delay divisioning technique, first the sufficient condition is presented when the delay interval is partitioned into two parts. Then, to reduce the conservatism, we extend this case to the general one, that is, we will partition the delay interval into arbitrary $m(m \geq 2)$ parts. In addition, Examples 1 and 2 have been provided to illustrate the effectiveness of this paper by comparing the result of Theorem 3 with some existing ones.

B. State Feedback Fuzzy Control

The previous section presents a new delay-dependent stability criterion for the fuzzy time-varying delay systems. In this section, our attention is focus on the design of state-feedback controller for the T–S fuzzy systems (3).

Theorem 4: Consider the delayed fuzzy systems in (3). Given scalars $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{i(2m+5)}$, d_0, d_m, $m \geq 2$, and τ, a state-feedback controller in the form of (1) is existing to ensure the closed-loop fuzzy system in (3) asymptotically stable, if there exist matrices $\hat{S}_N > 0$, $\hat{R}_N > 0$, $\hat{Q}_N > 0$, $N = 0, 1, \ldots, m$, X, $\hat{Z}_l$, $l = 1, 2, \ldots, m$, $\hat{P} > 0$, $\hat{Y} > 0$, and G_i satisfying the following inequalities for $i, j = 1, 2, \ldots, r$:

$$\Pi_{ii} + \Pi_{lii} < 0 \tag{23}$$

$$\Pi_{ij} + \Pi_{lij} + \Pi_{ji} + \Pi_{lji} < 0, \quad 1 \leq i < j \leq r \tag{24}$$

$$\begin{bmatrix} \hat{S}_l & \hat{Z}_l \\ \star & \hat{S}_l \end{bmatrix} \geq 0, \quad l = 1, \ldots, m \tag{25}$$

where

$$\begin{aligned}
\Pi_{ij} \triangleq{}& \sum_{N=0}^{m} \frac{\left(d_N^2 - d_{N-1}^2\right)^2}{4} W_{2m+5}^{\mathrm{T}} \hat{R}_N W_{2m+5} \\
&- \sum_{N=0}^{m} (\eta_N W_1 - W_{m+4+N})^{\mathrm{T}} \hat{R}_N (\eta_N W_1 - W_{m+4+N}) \\
&+ \sum_{N=0}^{m} \eta_N^2 W_{2m+5}^{\mathrm{T}} \hat{S}_N W_{2m+5} \\
&+ \sum_{N=0}^{m} \left(W_{N+1}^{\mathrm{T}} \hat{Q}_N W_{N+1} - W_{N+2}^{\mathrm{T}} \hat{Q}_N W_{N+2} \right) \\
&+ W_1^{\mathrm{T}} \hat{Y} W_1 + \mathrm{sym}\left(W_1^{\mathrm{T}} \hat{P} W_{2m+5} \right) \\
&- (W_1 - W_2)^{\mathrm{T}} \hat{S}_0 (W_1 - W_2) + (\tau - 1) W_{m+3}^{\mathrm{T}} \hat{Y} W_{m+3} \\
&+ \mathrm{sym}\left(\hat{M}_i \hat{W}_{Aij} \right) \\
\Pi_{lij} \triangleq{}& -\begin{bmatrix} W_{m+3} - W_{l+1} \\ W_{m+3} - W_{l+2} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \hat{S}_l & \hat{Z}_l \\ \star & \hat{S}_l \end{bmatrix} \begin{bmatrix} W_{m+3} - W_{l+1} \\ W_{m+3} - W_{l+2} \end{bmatrix} \\
&- \sum_{h=1,\ h\neq l}^{m} (W_{h+1} - W_{h+2})^{\mathrm{T}} \hat{S}_h (W_{h+1} - W_{h+2}) \\
\hat{W}_{Aij} \triangleq{}& \left[A_i X + B_i G_j \quad 0_{n,(m+1)n} \quad A_{di} X \quad 0_{n,(m+1)n} \quad -X \right] \\
\hat{M}_i \triangleq{}& \left[\lambda_{i1} I_n \quad \lambda_{i2} I_n \quad \ldots \quad \lambda_{i(2m+5)} I_n \right]^{\mathrm{T}}.
\end{aligned}$$

If the aforementioned conditions are feasible, the feedback gain matrices of the fuzzy systems can be constructed by

$$K_i = G_i X^{-1}, \quad i = 1, 2, \ldots, r. \tag{26}$$

Proof: For brevity, define $Z = X^{-\mathrm{T}}$, and introduce the following matrix:

$$E \triangleq \mathrm{diag}\{\underbrace{Z, Z, \ldots, Z}_{2m+5}\}.$$

Pre- and post-multiplying (23) and (24) by E and E^{T}, respectively, yields

$$E(\Pi_{ii} + \Pi_{lii})E^{\mathrm{T}} < 0$$

$$E\left(\Pi_{ij} + \Pi_{lij}\right)E^{\mathrm{T}} + E\left(\Pi_{ji} + \Pi_{lji}\right)E^{\mathrm{T}} < 0, \quad 1 \leq i < j \leq r$$

where

$$\begin{aligned}
&E\Pi_{ij}E^{\mathrm{T}} \\
&= \sum_{N=0}^{m} \frac{\left(d_N^2 - d_{N-1}^2\right)^2}{4} W_{2m+5}^{\mathrm{T}} Z\hat{R}_N Z^{\mathrm{T}} W_{2m+5} \\
&\quad + \sum_{N=0}^{m} \eta_N^2 W_{2m+5}^{\mathrm{T}} Z\hat{S}_N Z^{\mathrm{T}} W_{2m+5} + W_1^{\mathrm{T}} Z\hat{Y} Z^{\mathrm{T}} W_1 \\
&\quad - \sum_{N=0}^{m} (\eta_N W_1 - W_{m+4+N})^{\mathrm{T}} Z\hat{R}_N Z^{\mathrm{T}} (\eta_N W_1 - W_{m+4+N}) \\
&\quad + \sum_{N=0}^{m} \left(W_{N+1}^{\mathrm{T}} Z\hat{Q}_N Z^{\mathrm{T}} W_{N+1} - W_{N+2}^{\mathrm{T}} Z\hat{Q}_N Z^{\mathrm{T}} W_{N+2} \right) \\
&\quad + \mathrm{sym}\left(W_1^{\mathrm{T}} Z\hat{P} Z^{\mathrm{T}} W_{2m+5} \right) - (W_1 - W_2)^{\mathrm{T}} Z\hat{S}_0 Z^{\mathrm{T}} (W_1 - W_2) \\
&\quad + (\tau - 1) W_{m+3}^{\mathrm{T}} Z\hat{Y} Z^{\mathrm{T}} W_{m+3} + \mathrm{sym}\left(E\hat{M}_i \hat{W}_{Aij} E^{\mathrm{T}} \right) \\
&E\Pi_{lij}E^{\mathrm{T}} \\
&= -\begin{bmatrix} W_{m+3} - W_{l+1} \\ W_{m+3} - W_{l+2} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} Z\hat{S}_l Z^{\mathrm{T}} & Z\hat{Z}_l Z^{\mathrm{T}} \\ \star & Z\hat{S}_l Z^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} W_{m+3} - W_{l+1} \\ W_{m+3} - W_{l+2} \end{bmatrix} \\
&\quad - \sum_{h=1,\ h\neq l}^{m} (W_{h+1} - W_{h+2})^{\mathrm{T}} Z\hat{S}_h Z^{\mathrm{T}} (W_{h+1} - W_{h+2}).
\end{aligned}$$

Here, we define

$$\begin{aligned}
&S_N \triangleq Z\hat{S}_N Z^{\mathrm{T}}, \quad R_N \triangleq Z\hat{R}_N Z^{\mathrm{T}}, \quad Q_N \triangleq Z\hat{Q}_N Z^{\mathrm{T}} \\
&Y \triangleq Z\hat{Y} Z^{\mathrm{T}}, \quad P \triangleq Z\hat{P} Z^{\mathrm{T}}, \quad Z_l \triangleq Z\hat{Z}_l Z^{\mathrm{T}} \\
&M_i \triangleq E\hat{M}_i.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&E\hat{M}_i \hat{W}_{Aij} E^{\mathrm{T}} \\
&\quad = \left[\lambda_{i1} Z^{\mathrm{T}} \quad \lambda_{i2} Z^{\mathrm{T}} \quad \ldots \quad \lambda_{i(2m+5)} Z^{\mathrm{T}} \right]^{\mathrm{T}} \\
&\qquad \times \left[A_i X + B_i G_j \ 0_{n,(m+1)n} \ A_{di} X \ 0_{n,(m+1)n} \ -X \right] E^{\mathrm{T}} \\
&\quad = M_i \left[A_j + B_j K_k \ 0_{n,(m+1)n} \ A_{dj} \quad 0_{n,(m+1)n} \ -I_n \right], \\
&\qquad i, j, k = 1, 2, \ldots, r.
\end{aligned}$$

Therefore, we can see that the results obtained by pre- and post-multiplying (23) and (24) are equivalent to the inequalities (20) and (21). Similarly, pre- and post-multiplying (25) by $\mathrm{diag}\{Z, Z\}$ and $\mathrm{diag}\{Z^{\mathrm{T}}, Z^{\mathrm{T}}\}$, we can derive the inequality (22). Thus, according to the results of Theorem 3, the T–S fuzzy

TABLE I
MAXIMUM ALLOWABLE UPPER BOUND FOR DIFFERENT LOWER BOUND d_0

Method	$d_0 = 0.2$	$d_0 = 0.4$	$d_0 = 0.6$	$d_0 = 0.8$
$Lien, et\ al.$ [14]	0.7945	0.8487	0.9316	1.0325
$Tian, et\ al.$ [26]	1.141	1.150	1.172	1.209
$Souza, et\ al.$ [19]	1.1639	1.1734	1.1994	1.2532
$Theorem\ 3,\ m = 2$	1.2660	1.2831	1.2871	1.2623
$Theorem\ 3,\ m = 3$	1.2977	1.3054	1.2988	1.2672
$Theorem\ 3,\ m = 4$	1.3101	1.3116	1.2999	1.2674

delayed system with the above controller is asymptotically stable, and the proof is completed. ∎

IV. ILLUSTRATIVE EXAMPLES

In this section, we will present three examples to demonstrate the effectiveness and superiority of the proposed techniques in this paper. The first and second ones are used to illustrate our stability criterion is less conservative than the results developed in the literatures detailedly. The third one is provided to indicate applicability of our proposed method.

Example 1: Consider the continuous-time T–S fuzzy delayed system in (3) with two fuzzy rules, and the system matrices are given as follows:

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},\quad A_2 = \begin{bmatrix} -1.5 & 1 \\ 0 & -0.75 \end{bmatrix}$$
$$A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},\quad A_{d2} = \begin{bmatrix} -1 & 0 \\ 1 & -0.85 \end{bmatrix}.$$

Here, we set the system matrices $B_i = 0$.

The purpose here is to seek for the maximum allowable upper bound of time-varying delay under the given lower delay bound as well as the above fuzzy system is asymptotically stable. Table I shows the detailed comparison concerning achieved the upper delay bounds for different d_0 between the proposed Theorem 3 and [14], [19], and [26].

In order to show the merits of our proposed method, we plot the obtained upper bound of the time-varying delay for respective lower delay bounds, from which we can readily acquire the more detailed compared results.

Fig. 1 lucidly shows that our stability criterion can achieve the largest upper bound of the time-varying delay contrast with the existing results [14], [19], [26]. Meanwhile, we can conclude from Fig. 1 that the conservatism is reduced obviously with the delay interval partitioning fractions increasing. However, it should be noted that, although conservatism is reduced as the fraction becomes thinner, there is no evident improvement when the lower bound d_0 gets close to 0.6 or bigger.

Example 2: Consider the continuous-time T–S fuzzy system with the time-varying delay in (3) with two fuzzy rules, and the parameters of the system are given by

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},\quad A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}$$
$$A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},\quad A_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}.$$

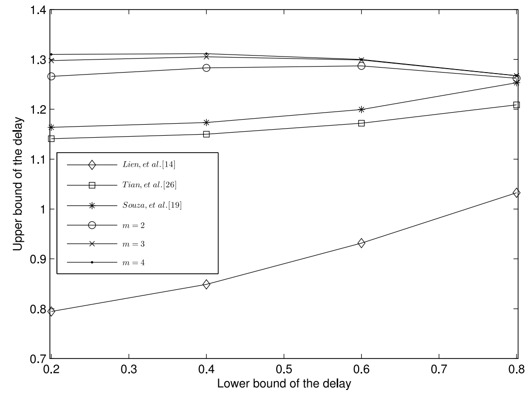

Fig. 1. Upper delay bound corresponding to the varying lower delay bound.

TABLE II
MAXIMUM ALLOWABLE UPPER BOUND

Method	Upper delay bound	Number of variables
$Tian\ and\ Peng$ [27]	1.5974	24
$Yoneyama$ [30]	1.6341	23
$Peng, et\ al.$ [17]	1.6341	15
$Theorem\ 3,\ m = 2$	1.7705	15
$Theorem\ 3,\ m = 4$	1.7770	23

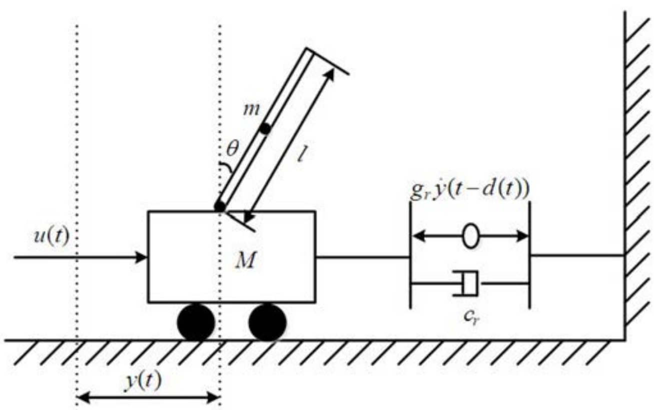

Fig. 2. Inverted pendulum on a cart with a delayed resonator.

Let us set $B_i = 0$ and consider the fuzzy system with a constant time delay, that is, the derivative of the time delay is set to be zero, i.e., $\dot{d}(t) = 0$. Table II shows the results of the maximum allowable delay bound generated by related references [17], [27], [30] and our method. It is seen from Table II our method succeeds in achieving the biggest allowable upper delay bound with respect to other references. Compared with the methods in [17], [27], and [30], our method requires the fewer free variables and performs well. Furthermore, the number of unknown variables used in the computation is the denotation of the computation burden.

Example 3: In this example, we consider the inverted pendulum system investigated in [13]. The schematic diagram of this pendulum system is denoted in Fig. 2. To design a fuzzy controller, we need establish a T–S fuzzy model for the corresponding nonlinear system. Consequently, in the first place, we will construct the T–S fuzzy model of the pendulum system by some approximation techniques, and then stabilize the inverted pendulum system by utilizing our designed controller.

TABLE III
COMPARISON OF UPPER BOUNDS AND CONTROLLER FEEDBACK GAINS FOR DIFFERENT CASES

Method	d_0	$\bar{d}$	K_1	K_2
$Theorem\ 4,\ m=2$	0.1	1.4527	[0.0013 30.2859 7.5837 5.1625]	[0.0011 29.5536 7.3600 4.4054]
	0.3	1.5321	[0.0026 31.6369 7.6595 5.3417]	[0.0022 30.6965 7.4303 4.5661]
	0.5	1.3224	[0.0028 29.7307 7.4063 5.0375]	[0.0026 29.0849 7.2264 4.3056]
$Theorem\ 4,\ m=3$	0.1	1.4575	[0.0134 30.7909 7.6601 5.2958]	[0.0112 30.0020 7.4255 4.4940]
	0.3	1.5797	[0.0059 31.5425 7.6834 5.3664]	[0.0052 30.6082 7.4267 4.5654]
	0.5	1.4074	[0.0054 30.7601 7.6066 5.2111]	[0.0048 29.9372 7.3605 4.4352]
$Theorem\ 4,\ m=4$	0.1	1.5456	[0.1696 31.2491 7.7618 5.4203]	[0.1422 30.2531 7.4861 4.5755]
	0.3	1.5985	[0.0103 31.8430 7.6205 5.4085]	[0.0088 30.8357 7.3804 4.6171]
	0.5	1.4615	[0.0107 30.8056 7.4986 5.2148]	[0.0095 29.9472 7.2809 4.4471]

The parameters of the pendulum system are presented as follows.

M	Mass of the cart, 1.378 kg.
m	Mass of the pendulum, 0.051 kg.
l	Length of the pendulum, 0.325 m.
g_r	Coefficient of the delayed resonator, 0.7 kg/s.
g	Acceleration due to gravity, 9.8 m/s^2.
c_r	Coefficient of the damper, 5.98 kg/s.
$\theta(t)$	Angle the pendulum makes with the top vertical.
$y(t)$	Displacement of the cart.
$d(t)$	Time-varying delay.
$u(t)$	Force applied to the cart.

For notation simplicity, the notation "(t)" in some places will be omitted. We assume that the pendulum can be modeled as a thin rod. By Newton's law, we arrive at the equations of motion for the system

$$\begin{aligned} M\frac{d^2y}{dt^2} + m\frac{d^2}{dt^2}(y + l\sin\theta) &= u - F_r \\ m\frac{d^2}{dt^2}(y + l\sin\theta)l\cos\theta &= mgl\sin\theta \end{aligned}$$

where $F_r(t) = g_r\dot{y}(t - d(t)) + c_r\dot{y}(t)$ indicates the force of the damper and delayed resonator. According to the above equations, and the choice of the state variables $x_1 = y$, $x_2 = \theta$, $x_3 = \dot{y}$, and $x_4 = \dot{\theta}$, the achieved system state-space equations are as follows:

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= \frac{-mg\sin x_2}{M\cos x_2} - \frac{c_r x_3 + g_r x_3(t - d(t)) - u}{M} \\ \dot{x}_4 &= \frac{(M+m)g\sin x_2}{Ml\cos^2 x_2} + \frac{x_4^2\sin x_2}{\cos x_2} \\ &\quad + \frac{c_r x_3 + g_r x_3(t - d(t)) - u}{Ml\cos x_2}. \end{aligned}$$

In order to receive our fuzzy controller, we will construct the following T–S fuzzy model which represents the aforementioned inverted pendulum system.

A. Plant Form

Rule 1: IF x_2 is about 0 rad, THEN $\dot{x}(t) = A_1x(t) + A_{d1}x(t - d(t)) + B_1u(t)$.

Rule 2: IF x_2 is near γ $(0 < |\gamma| < 1.57$ rad), THEN $\dot{x}(t) = A_2x(t) + A_{d2}x(t - d(t)) + B_2u(t)$

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mg}{M} & -\frac{c_r}{M} & 0 \\ 0 & \frac{(M+m)g}{Ml} & \frac{c_r}{Ml} & 0 \end{bmatrix}$$

$$A_{d1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{g_r}{M} & 0 \\ 0 & 0 & \frac{g_r}{Ml} & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{Ml} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mg\beta}{M\alpha} & -\frac{c_r}{M} & 0 \\ 0 & \frac{(M+m)g\beta}{Ml\alpha^2} & \frac{c_r}{Ml\alpha} & 0 \end{bmatrix}$$

$$A_{d2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{g_r}{M} & 0 \\ 0 & 0 & \frac{g_r}{Ml\alpha} & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{Ml\alpha} \end{bmatrix}$$

where $x(t) = [x_1^{\mathrm{T}}(t) \;\; x_2^{\mathrm{T}}(t) \;\; x_3^{\mathrm{T}}(t) \;\; x_4^{\mathrm{T}}(t)]^{\mathrm{T}}$, set $|\gamma| = 0.52$ rad, $\alpha = \cos\gamma$, and $\beta = (\sin\gamma)/\gamma$. The fuzzy basis functions $h_1(x_2(t))$ and $h_2(x_2(t))$ are chosen as triangular ones, the expression are

$$h_1(x_2(t)) = 1 - \frac{|x_2(t)|}{|\gamma|}, \quad h_2(x_2(t)) = \frac{|x_2(t)|}{|\gamma|}.$$

First, we make a comparison concerning the upper bounds obtained from the inverted pendulum system under the different lower bound d_0. According to Table III which lists the upper bounds and the fuzzy controller gains, it can be concluded that the allowable upper bound of the controller constructed successfully by using Theorem 4 is enhanced along with the number of partitioned segments m raising. That is, the conservatism of our approach is decreased when the number of fractioning is increased. Meanwhile, the unknown variables increase. Therefore, there is a tradeoff between the conservatism reduction and the computation complexity.

To further illustrate the effectiveness of our proposed method, we set the partitioning fractions $m = 2$, $d_0 = 0.1$,

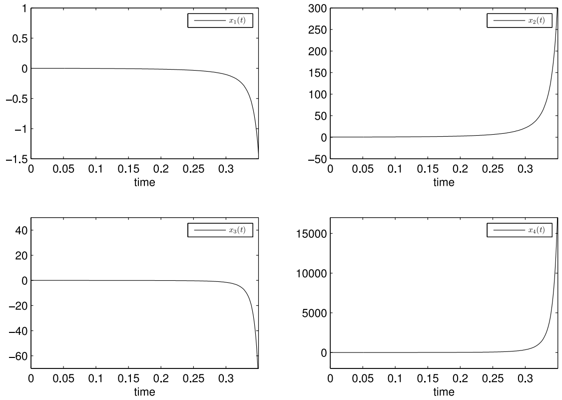

Fig. 3. States of the original system without control.

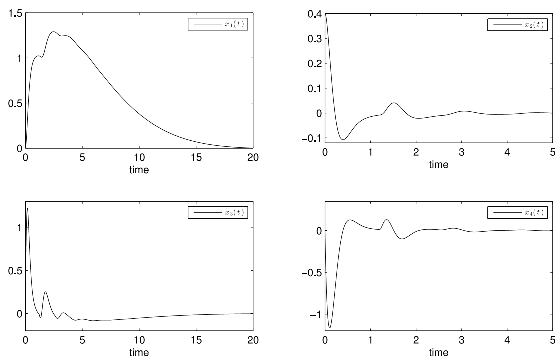

Fig. 4. States of the controlled fuzzy system.

and $\tau = 0.3$, and the initial condition is assumed to be $x(t) = [0 \quad 0.4 \quad 0 \quad 0]^{\mathrm{T}}$. The dynamic behavior of the original system is presented in Fig. 3, from which we learn that original inverted pendulum system is unstable. By applying Theorem 4, we stabilize the fuzzy system with the allowable upper bound of time-varying delay $\bar{d} = 1.3894$. Letting $\bar{d} = 1.2$, we find the following solutions satisfying the conditions of Theorem 4:

$$G_1 = \begin{bmatrix} -0.0265 & 0.0132 & -0.0291 & 0.0370 \end{bmatrix}$$
$$G_2 = \begin{bmatrix} -0.0272 & 0.0189 & -0.0301 & 0.0093 \end{bmatrix}$$
$$X = \begin{bmatrix} 0.0664 & -0.0011 & 0.0008 & -0.0011 \\ 0.0000 & 0.0017 & -0.0035 & -0.0040 \\ -0.0053 & 0.0018 & 0.0084 & -0.0108 \\ 0.0014 & -0.0109 & 0.0020 & 0.0498 \end{bmatrix}$$
$$K_1 = \begin{bmatrix} 0.0983 & 30.0079 & 7.7666 & 4.8405 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} 0.0851 & 29.4028 & 7.5489 & 4.1888 \end{bmatrix}.$$

According to the results received as above, the fuzzy controller constructed in (1) is given by

$$\begin{aligned} u(t) = {} & h_1(x_2(t))\begin{bmatrix} 0.0983 & 30.0079 & 7.7666 & 4.8405 \end{bmatrix}x(t) \\ & + h_2(x_2(t))\begin{bmatrix} 0.0851 & 29.4028 & 7.5489 & 4.1888 \end{bmatrix}x(t). \end{aligned}$$

Combined with the above controller, the obtained fuzzy state-feedback system makes the closed-loop states converge to zero, as is shown in Fig. 4.

V. CONCLUSION

In this paper, the problem of stability and stabilization for continuous-time T–S fuzzy systems with the time-varying delay has been investigated. Based on the delay partitioning method and the reciprocally convex approach, a new sufficient condition which reduces conservatism remarkably compared with quite a few existed results, has been developed. Then, the desired PDC controller is designed to stabilize the closed-loop fuzzy delayed systems. Finally, three illustrative examples are proposed to demonstrate the effectiveness and advantages of the results presented in this paper. The above results developed could be further extended to reduced-order fuzzy systems with the time-varying delay in our future work.

REFERENCES

[1] J. Chen, "Fuzzy singularly perturbed modeling and composite controller design for nonlinear multiple time-scale systems with time-delay," *Fuzzy Sets Syst.*, vol. 254, pp. 142–156, Nov. 2014.

[2] S. F. Derakhshan, A. Fatehi, and M. G. Sharabiany, "Nonmonotonic observer-based fuzzy controller designs for discrete time T–S fuzzy systems via LMI," *IEEE Trans. Cybern.*, vol. 44, no. 12, pp. 2557–2567, Dec. 2014.

[3] Z. Feng, J. Lam, and G.-H. Yang, "Optimal partitioning method for stability analysis of continuous/discrete delay systems," *Int. J. Robust Nonlin. Control*, vol. 25, no. 4, pp. 559–574, Mar. 2015.

[4] Q. Gao, G. Feng, Z. Xi, Y. Wang, and J. Qiu, "Robust H_∞ control of T–S fuzzy time-delay systems via a new sliding-mode control scheme," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 2, pp. 459–465, Apr. 2014.

[5] H. Guo, J. Qiu, H. Tian, and H. Gao, "Fault detection of discrete-time T–S fuzzy affine systems based on piecewise Lyapunov functions," *J. Franklin Inst.*, vol. 351, no. 7, pp. 3633–3650, 2014.

[6] T. M. Guerra and L. Vermeiren, "LMI-based relaxed nonquadratic stabilization conditions for nonlinear systems in the Takagi–Sugeno's form," *Automatica*, vol. 40, no. 5, pp. 823–829, 2004.

[7] N. Kazachek, V. Lokhin, S. Manko, and M. Romanov, "Research of periodic oscillations in control systems with fuzzy controllers," *Int. J. Innov. Comput. Inf. Control*, vol. 11, no. 3, pp. 985–997, 2015.

[8] F. Li, P. Shi, L. Wu, and X. Zhang, "Fuzzy-model-based $\mathcal{D}$-stability and nonfragile control for discrete-time descriptor systems with multiple delays," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 4, pp. 1019–1025, Aug. 2014.

[9] H.-K. Lam, "LMI-based stability analysis for fuzzy-model-based control systems using artificial T–S fuzzy model," *IEEE Trans. Fuzzy Syst.*, vol. 19, no. 3, pp. 505–513, Jun. 2011.

[10] H.-K. Lam and J.-C. Lo, "Output regulation of polynomial-fuzzy-model-based control systems," *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 2, pp. 262–274, Apr. 2013.

[11] T.-F. Li, J. Zhao, and G. M. Dimirovski, "Stability and $\mathcal{L}$-gain analysis for switched neutral systems with mixed time-varying delays," *J. Franklin Inst.*, vol. 348, no. 9, pp. 2237–2256, 2011.

[12] D. H. Lee, J. B. Park, and Y. H. Joo, "Approaches to extended non-quadratic stability and stabilization conditions for discrete-time Takagi–Sugeno fuzzy systems," *Automatica*, vol. 47, no. 3, pp. 534–538, Mar. 2011.

[13] M. Landry, S. A. Campbell, K. Morris, and C. O. Aguilar, "Dynamics of an inverted pendulum with delayed feedback control," *SIAM J. Appl. Dyn. Syst.*, vol. 4, no. 2, pp. 333–351, 2005.

[14] C. H. Lien, K. W. Yu, W. D. Chen, Z. L. Wan, and Y. J. Chung, "Stability criteria for uncertain Takagi–Sugeno fuzzy systems with interval time-varying delay," *IET Control Theory Appl.*, vol. 1, no. 3, pp. 764–769, May 2007.

[15] A. E. Ougli and B. Tidhaf, "Optimal type-2 fuzzy adaptive control for a class of uncertain nonlinear systems using an LMI approach," *Int. J. Innov. Comput. Inf. Control*, vol. 11, no. 3, pp. 851–863, 2015.

[16] P. G. Park, J. W. Ko, and C. Jeong, "Reciprocally convex approach to stability of systems with time-varying delays," *Automatica*, vol. 47, no. 1, pp. 235–238, 2011.

[17] C. Peng, D. Yue, T.-C. Yang, and E. G. Tian, "On delay-dependent approach for robust stability and stabilization of T–S fuzzy systems with constant delay and uncertainties," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 5, pp. 1143–1156, Oct. 2009.

[18] J.-T. Pan, S.-M. Fei, A. Jaadri, and T. M. Guerra, "Nonquadratic stabilisation of continuous T–S models: LMI solution for a local approach," *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 3, pp. 594–602, Jun. 2012.
[19] F. O. Souza, V. C. S. Campos, and R. M. Palhares, "On delay-dependent stability conditions for Takagi–Sugeno fuzzy systems," *J. Franklin Inst.*, vol. 351, no. 7, pp. 3707–3718, 2014.
[20] J. Sun, G. P. Liu, J. Chen, and D. Rees, "Improved delay-range-dependent stability criteria for linear systems with time-varying delays," *Automatica*, vol. 46, no. 2, pp. 466–470, 2010.
[21] H. Shao, G. Miao, and Z. Zhang, "State feedback control design for a networked control model of systems with two additive time-varying delays," *Int. J. Innov. Comput. Inf. Control*, vol. 11, no. 4, pp. 1457–1469, 2015.
[22] X. Su, L. Wu, and P. Shi, "Sensor networks with random link failures: Distributed filtering for T–S fuzzy systems," *IEEE Trans. Ind. Informat.*, vol. 9, no. 3, pp. 1739–1750, Aug. 2013.
[23] X. Su, L. Wu, P. Shi, and Y.-D. Song, "H_∞ model reduction of Takagi–Sugeno fuzzy stochastic systems," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 42, no. 6, pp. 1574–1585, Dec. 2012.
[24] X. Su, L. Wu, P. Shi, and C. L. P. Chen, "Model approximation for fuzzy switched systems with stochastic perturbation," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 5, pp. 1458–1473, Oct. 2015.
[25] X. Su, P. Shi, L. Wu, and M. V. Basin, "Reliable filtering with strict dissipativity for T–S fuzzy time-delay systems," *IEEE Trans. Cybern.*, vol. 44, no. 12, pp. 2470–2483, Dec. 2014.
[26] E. Tian, D. Yue, and Y. Zhang, "Delay-dependent robust H_∞ control for T–S fuzzy system with interval time-varying delay," *Fuzzy Sets Syst.*, vol. 160, no. 12, pp. 1708–1719, 2009.
[27] E. Tian and C. Peng, "Delay dependent stability analysis and synthesis of uncertain T–S fuzzy systems with time-varying delay," *Fuzzy Sets Syst.*, vol. 157, no. 4, pp. 544–559, 2006.
[28] L. Wu and D. W. C. Ho, "Fuzzy filter design for Itô stochastic systems with application to sensor fault detection," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 1, pp. 233–242, Feb. 2009.
[29] X. Xie, D. Yue, T. Ma, and X. Zhu, "Further studies on control synthesis of discrete-time T–S fuzzy systems via augmented multi-indexed matrix approach," *IEEE Trans. Cybern.*, vol. 44, no. 12, pp. 2784–2791, Dec. 2014.
[30] J. Yoneyama, "New delay-dependent approach to robust stability and stabilization for Takagi–Sugeno fuzzy time-delay systems," *Fuzzy Sets Syst.*, vol. 158, no. 20, pp. 2225–2237, 2007.
[31] Y. Zhao, H. Gao, J. Lam, and B. Du, "Stability and stabilization of delayed T–S fuzzy systems: A delay partitioning approach," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 4, pp. 750–762, Aug. 2009.
[32] H. Zhang and Z. Liu, "Stability analysis for linear delayed systems via an optimally dividing delay interval approach," *Automatica*, vol. 47, no. 9, pp. 2126–2129, 2011.
[33] J. Zhang, P. Shi, and Y. Xia, "Fuzzy delay compensation control for T–S fuzzy systems over network," *IEEE Trans. Cybern.*, vol. 43, no. 1, pp. 259–268, Feb. 2013.
[34] H. Zhang and J. Wang, "State estimation of discrete-time Takagi–Sugeno fuzzy systems in a network environment," *IEEE Trans. Cybern.*, vol. 45, no. 8, pp. 1525–1536, Aug. 2015.
[35] D. Zhang, W. J. Cai, L. H. Xie, and Q.-G. Wang, "Nonfragile distributed filtering for T–S fuzzy systems in sensor networks," *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 5, pp. 1883–1890, Oct. 2015.

Xiaojie Su (S'12–M'13) was born in Henan, China, in 1985. He received the B.S. degree in automation from Jiamusi University, Jiamusi, China, in 2008, and the M.E. and Ph.D. degrees in control theory and control engineering from the Harbin Institute of Technology, Harbin, China, in 2010 and 2013, respectively.

He is currently with the College of Automation, Chongqing University, Chongqing, China. His current research interests include Takagi–Sugeno fuzzy systems, optimal filtering, optimal controller design, and model reduction.

Dr. Su currently serves as an Associate Editor for a number of journals, including the IEEE ACCESS, *Neurocomputing*, and *Information Sciences and Computing*.

Hongying Zhou was born in Chongqing, China, in 1990. She received the B.S. degree in automation from Chongqing University, Chongqing, China, in 2013, and the M.E. degree in control science and engineering from the School of Automation, Chongqing University, in 2016.

Her current research interests include Takagi–Sugeno fuzzy systems, model reduction, and fuzzy controller design.

Yong-Duan Song was born in 1962, in Chongqing, China. He received the Ph.D. degree in electrical and computer engineering from Tennessee Technological University, Cookeville, TN, USA, in 1992.

He was a tenured Full Professor with North Carolina A&T State University, Greensboro, NC, USA, from 1993 to 2008, a Langley Distinguished Professor from 2005 to 2008, one of the six Langley Distinguished Professors, and the Founding Director of Cooperative Systems with the National Institute of Aerospace, Hampton, VA, USA. He is currently the Dean of the School of Automation, and the Founding Director of the Institute of Smart Systems and Renewable Energy, Chongqing University, Chongqing, China. His current research interests include intelligent systems, guidance navigation and control, bioinspired adaptive and cooperative systems, rail traffic control and safety, and smart grid.

Prof. Song was a recipient of several competitive research awards from the National Science Foundation, the National Aeronautics and Space Administration, the U.S. Air Force Office, the U.S. Army Research Office, and the U.S. Naval Research Office. He has served as an Associate Editor/Guest Editor for several prestigious scientific journals.