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Stability Analysis of Polynomial Fuzzy Systems with Time-Delay via Sum Of Squares (SOS) Approach

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***Résumé*—This paper presents a sum of squares (SOS) approach to stability analysis of polynomial fuzzy systems with time-delay. Based on a novel polynomial Lyapunov-Krasovskii Functional, stability conditions are presented in terms of sum of squares in order to reduce the conservatism. The proposed SOS-based framework is more innovative and effective than the existing linear matrix inequality (LMI). The SOS designed conditions can be solved via the recent developed toolbox of Matlab, SOSTOOLS V3.00. Numerical example is given to illustrate the utility of the proposed approach.**

***Keywords*—Polynomial fuzzy systems, time-delay, stability, polynomial Lyapunov-Krasovskii Functional, sum of squares (SOS).**

I. Introduction

During this last decade, polynomial fuzzy models have been received a large deal of attention. The crucial difference between this class of models and the well-known Takagi-Sugeno fuzzy models appears in the consequence part representation [13]-[16]. The T-S fuzzy model has linear model in consequence part while the recent polynomial fuzzy model presents polynomial model in its consequence part [17]. Polynomial fuzzy models can represent the dynamic of nonlinear systems in a more general way than T-S fuzzy models. So, polynomial fuzzy systems are more general than T-S fuzzy systems and allow to obtain more relaxed stability conditions than the existing LMI results by using a polynomial Lyapunov function.

On the other hand, it is well-known that time delay is a common and complex phenomenon in various industrial processes. It often appears in a wide class of systems, such as transportation systems, rolling mill systems, network systems and communication systems [6]. It is usually a source of instability and degraded performance of systems. Therefore, recently, there have been many of research activities on time-delay systems [3]. In the last few years, polynomial fuzzy systems with time delay have received a great attention and become a topical problem.

For example, in 2012, W. Li and W. Wang have proposed SOS stability conditions with guaranteed cost control for polynomial fuzzy systems with state time delay using the following Lyapunov-Krasovskii Functional :

$V(x(t)) = x(t)^T P(x)x(t) + \int_{t-\tau}^{t} x(\alpha)^T Sx(\alpha)d\alpha$

It is generally known that delay-dependent results are usually more relaxed than delay-independent ones, especially when the size of the delay is small. For that, our objective in this paper is to establish new stability conditions using the following Lyapunov-Krasovskii Functional :

$$\begin{aligned} V(x(t)) = \;& x(t)^T P(x)x(t) + \int_{t-\tau(t)}^{t} x(\alpha)^T S(x)x(\alpha)d\alpha \\ & + \int_{-\overline{\tau}}^{0}\int_{t+\sigma}^{t} \dot{x}(\alpha)^T Z(x)\dot{x}(\alpha)d\alpha d\sigma \end{aligned}$$

The obtained results are delay-dependent, which reduce the conservatism of delay-independent results. The main purpose of this work is to present a novel technique in order to reduce the conservatism and decrease computational complexity by using a LKF. The design stability conditions are formulated in terms of SOS.

The rest of the paper is organized as follows. Section II introduces a polynomial fuzzy model with time varying delay and presents also a review of some results to be used in this paper. New delay dependent stability conditions are established in section III. Section IV presents a numerical example to demonstrate the effectiveness and the viability of the proposed method. Some conclusions are drawn in section V.

II. Problem Formulation

Consider a nonlinear time-delay system that can be represented by the following polynomial fuzzy time-delay model :

Model rule i : $(i = 1, 2, \cdots, r)$

If θ_1 is μ_{i1} and $\cdots$ and θ_p is μ_{ip}

Then

$$\begin{aligned} \dot{x}(t) &= A_i(x)x(t) + A_{\tau i}(x)x(t-\tau(t)) \\ x(t) &= \psi(t), t \in [-\overline{\tau}, 0], \end{aligned} \tag{1}$$

where $\theta_j(x(t))(j = 1, \cdots, p)$ are the premise variables. $\mu_{ij}(i = 1, \cdots, r, j = 1, \cdots, p)$ are the fuzzy sets. r denotes the number of IF-THEN rules. $\psi(t)$ is the initial conditions. $x(t) \in \Re^n$ is the state vector. $A_i(x)$, $A_{\tau i}(x)$ are polynomial matrices in $x(t)$ with appropriate dimensions.

Time delay, $\tau(t)$, is a time-varying continuous function that satisfies :

$$0 \leq \tau(t) \leq \overline{\tau}, \dot{\tau}(t) \leq \beta \tag{2}$$

By using singleton fuzzifier, the common used center-average defuzzifier and product interference, fuzzy model (1) can be represented as :

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} h_i(\theta(x(t)))[A_i(x)x(t) + A_{\tau i}(x)x(t-\tau(t))] \\ x(t) &= \psi(t), t \in [-\overline{\tau}, 0], \end{aligned} \tag{3}$$

where $\theta(x(t)) = [\theta_1(x(t)), \cdots, \theta_p(x(t))]$ and $v_i(\theta(x(t))) : \Re^p \rightarrow [0,1], i = 1, \cdots, r,$ is the membership function, $v_i(\theta(x(t))) = \prod_{j=1}^{p} \mu_{ij}(\theta_j(x(t)))$

Denotes $h_i(\theta(x(t))) = \frac{v_i(\theta(x(t)))}{\sum_{i=1}^{r} v_i(\theta(x(t)))}$.

It should be noted from the properties of membership functions that the following relations hold :

$$\sum_{i=1}^{r} v_i(\theta(x(t))) > 0, \quad v_i(\theta(x(t))) \geq 0, \quad i = 1, \cdots, r.$$

Hence,

$$h_i(\theta(x(t))) \geq 0 \ , \ \sum_{i=1}^{r} h_i(\theta(x(t))) = 1$$

In order to obtain the main results in this paper, the following lemmas are proposed

Lemma 1 : [15] : Let $f(x(t))$ be a polynomial in $x(t) \in \Re^n$. $f(x(t))$ is an SOS if there exists a positive semi-definite matrix $P(x)$ such that

$$f(x(t)) = x^T P(x) x \tag{4}$$

It is clear that $f(x(t))$ being an SOS implies that $f(x(t)) > 0$

Lemma 2 : [18]-[5]-[9] Consider a negative definite matrix $\Pi < 0$.
Given X a matrix of appropriate dimension such that $X^T \Pi X < 0$, then, $\exists \lambda \in \Re^+$ such that

$$X^T \Pi X \leq -2\lambda X - \lambda^2 \Pi^{-1} \tag{5}$$

From now, for brevity, we will use h_i to denote $h_i(\theta(x(t)))$.

The purpose of this paper is to derive stability for polynomial fuzzy systems with time-delay. We will establish delay-dependent SOS conditions for this class of systems. We will show that our delay-dependent results are less conservative than existing ones in [8]. And in order to solve these problems, we use the recent developed SOSTOOLS [12]-[11].

III. Main results

In the sequel, we will present our main results concerning the stability of polynomial fuzzy systems with time-delay . We consider below system (6), that is

$$\dot{x}(t) = \sum_{i=1}^{r} h_i[A_i(x)x(t) + A_{\tau i}(x)x(t-\tau(t))] \tag{6}$$

Theorem 1 : System (6) is asymptotically stable, if there exist some positive polynomial matrices $P(x), S(x)$ and $Z(x)$ satisfying the following conditions for $i = 1, 2, .., r$

$$P(x) - \varepsilon_1(x) I \quad is \quad SOS \quad matrix \tag{7}$$

$$S(x) - \varepsilon_2(x) I \quad is \quad SOS \quad matrix \tag{8}$$

$$Z(x) - \varepsilon_3(x) I \quad is \quad SOS \quad matrix \tag{9}$$

$$-\Phi_i(x) - \varepsilon_4(x) I \quad is \quad SOS \quad matrix \tag{10}$$

where :

$$\Phi_i(x) = \begin{bmatrix} \varphi_i(x) & P(x)A_{\tau i}(x) & A_i(x)^T Z(x) \\ * & -(1-\beta)S(x) & A_{\tau i}(x)^T Z(x) \\ * & * & \frac{-1}{\overline{\tau}} Z(x) \end{bmatrix} \tag{11}$$

$$\varphi_i(x) = P(x)A_i(x) + A_i(x)^T P(x) + S(x) + \sum_{k=1}^{n} \frac{\partial P(x)}{\partial x_k} (A_i^k(x) x)$$

$\varepsilon_i(x)$ are nonnegative polynomials such that $\varepsilon_i(x) \geq 0, \ \forall x \in \Re$ $(i = 1, ..., 4)$
$A_i^k(x)$ denotes the kth row of $A_i(x)$

Proof : Choose the LKF as

$$\begin{aligned} V(x(t)) = \ & x(t)^T P(x) x(t) + \int_{t-\tau(t)}^{t} x(\alpha)^T S(x) x(\alpha) d\alpha \\ & + \int_{-\overline{\tau}}^{0} \int_{t+\sigma}^{t} \dot{x}(\alpha)^T Z(x) \dot{x}(\alpha) d\alpha d\sigma \end{aligned} \tag{12}$$

The time derivative of this LKF (12) along the trajectory of system (6) is obtained as

$$\begin{aligned} \dot{V}(x(t)) = \ & \dot{x}(t)^T P(x) x(t) + x(t)^T P(x) \dot{x}(t) \\ & + x(t)^T \dot{P}(x) x(t) + x(t)^T S(x) x(t) \\ & -(1-\dot{\tau}(t)) x(t-\tau(t))^T S(x) x(t-\tau(t)) \\ & + \overline{\tau} \dot{x}(t)^T Z(x) \dot{x}(t) - \int_{t-\overline{\tau}}^{t} \dot{x}(s)^T Z(x) \dot{x}(s) ds \end{aligned} \tag{13}$$

We have

$$0 \leq \tau(t) \leq \overline{\tau} \quad , \quad \dot{\tau}(t) \leq \beta \tag{14}$$

$$\begin{aligned} \dot{P}(x) &= \sum_{k=1}^{n} \frac{\partial P(x)}{\partial x_k} \dot{x}_k \\ &= \sum_{i=1}^{r} h_i \sum_{k=1}^{n} \frac{\partial P(x)}{\partial x_k} (A_i^k(x) x) \end{aligned} \tag{15}$$

We obtain following equation (16) :

$$\begin{aligned}\dot{V}(x(t)) \leq & \sum_{i=1}^{r} h_i[(x(t)^T A_i(x)^T \\ & + x(t-\tau(t))^T A_{\tau i}(x)^T)P(x)x(t) \\ & + x(t)^T P(x)(A_i(x)x(t) + A_{\tau i}(x)x(t-\tau(t)) \\ & + x(t)^T (\sum_{k=1}^{n} \frac{\partial P(x)}{\partial x_k}(A_i^k(x)x))x(t)] \\ & + x(t)^T S(x)x(t) - (1-\beta)x(t-\tau(t))^T S(x)x(t-\tau(t)) \\ & + \overline{\tau}\dot{x}(t)^T Z(x)\dot{x}(t) - \int_{t-\overline{\tau}}^{t} \dot{x}(s)^T Z(x)\dot{x}(s)ds\end{aligned} \quad (16)$$

As it is shown in [1]

$$\dot{x}(t)^T Z(x)\dot{x}(t) \leq \sum_{i=1}^{r} h_i \eta(t)^T \begin{bmatrix} A_i(x)^T Z(x) A_i(x) & A_i(x)^T Z(x) A_{\tau i}(x) \\ A_{\tau i}(x)^T Z(x) A_i(x) & A_{\tau i}(x)^T Z(x) A_{\tau i}(x) \end{bmatrix} \eta(t) \quad (17)$$

where : $\eta(t)^T = [x(t)^T, x(t-\tau(t))^T]$

$$\begin{aligned}\dot{V}(x(t)) \leq & \sum_{i=1}^{r} h_i[x(t)^T A_i(x)^T P(x)x(t) \\ & + x(t-\tau(t))^T A_{\tau i}(x)^T P(x)x(t) + x(t)^T P(x)A_i(x)x(t) \\ & + x(t)^T P(x)A_{\tau i}(x)x(t-\tau(t)) \\ & + x(t)^T \sum_{k=1}^{n} \frac{\partial P(x)}{\partial x_k}(A_i^k(x)x)x(t) + x(t)^T S(x)x(t) \\ & + \eta(t)^T \overline{\tau} \begin{bmatrix} A_i(x)^T Z(x) A_i(x) & A_i(x)^T Z(x) A_{\tau i}(x) \\ A_{\tau i}(x)^T Z(x) A_i(x) & A_{\tau i}(x)^T Z(x) A_{\tau i}(x) \end{bmatrix} \eta(t) \\ & - (1-\beta)x(t-\tau(t))^T S(x)x(t-\tau(t))] - \int_{t-\overline{\tau}}^{t} \dot{x}(s)^T Z(x)\dot{x}(s)ds\end{aligned} \quad (18)$$

Let $\tilde{\Phi}_i(x)$ be a polynomial matrix that

$$\tilde{\Phi}_i(x) = \begin{bmatrix} \tilde{\Phi}_i^{11}(x) & P(x)A_{\tau i}(x) + \overline{\tau}A_i(x)^T Z(x) A_{\tau i}(x) \\ * & -(1-\beta)S(x) + \overline{\tau}A_{\tau i}(x)^T Z(x) A_{\tau i}(x) \end{bmatrix} \quad (19)$$

where

$$\tilde{\Phi}_i^{11}(x) = P(x)A_i(x) + A_i(x)^T P(x) + S(x) + \sum_{k=1}^{n} \frac{\partial P(x)}{\partial x_k}(A_i^k(x)x) + \overline{\tau}A_i(x)^T Z(x) A_i(x)$$

Therefore

$$\dot{V}(x(t)) \leq \sum_{i=1}^{r} h_i \eta(t)^T \tilde{\Phi}_i(x) \eta(t) - \int_{t-\overline{\tau}}^{t} \dot{x}(s)^T Z(x)\dot{x}(s)ds \quad (20)$$

Thus, by applying Schur complement,we obtain theorem1.

Remark 1 : Our method provides a less conservative result than other result which has been recently proposed in [8]. In next paragraph, a numerical example will be given to demonstrate numerically this point.

Remark 2 : When $A_i(x), A_{\tau i}(x)$ and $P(x)$ are constant matrices, the polynomial fuzzy system with time-delay (3) is the same as the well known T-S fuzzy model. Thus, the suggested SOS approach to polynomial fuzzy models contains the existing LMI approaches for T-S fuzzy models. So, the SOS framework for polynomial fuzzy systems reduces more the conservatism and affords more relaxed stability results.

IV. Illustrative Example

In order to demonstrate the effectiveness and the merits of the achieved results in this paper, let us consider the polynomial fuzzy system with a varying time-delay (21) :

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(z(t))[A_i(x)x(t) + A_{\tau i}(x)x(t-\tau(t))] \quad (21)$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -5 & -1 \end{bmatrix} \quad (22)$$

$$A_{\tau 1} = \begin{bmatrix} 0 & 0.1 \\ -0.2 & -0.1 \end{bmatrix}, A_{\tau 2} = \begin{bmatrix} 0 & 0.1 \\ -0.4 & -0.1 \end{bmatrix} \quad (23)$$

The membership functions are defined by

$$h_1(x_2) = 1 - \sin x_2 ; \quad h_2(x_2) = 1 - h_1(x_2)$$

The varying time-delay is given as follows :

$$\tau(t) = 1.10 + 0.75 \sin t$$

It is clear that the above model is submitted to a time-varying delay. Therefore, we can apply results of theorem (1). This example shows the less conservativeness of our approach compared with existing delay-independent SOS stability results established in [8]. The aim is to find a value of τ that ensures the stability of the polynomial fuzzy system.

Note that delay-independent conditions proposed in [8] fail to give a feasible solution for this example. However, by using the SOSTOOLS of Matlab [12], a feasible solution of theorem 1 can be obtained for $\tau = 1$. So, the approach proposed in this paper provides more relaxed stability conditions than those given in [8].

Any quadratic Lyapunov function for the above polynomial fuzzy system with time-delay do not exist, for all values of τ. But all system's trajectories converge to zero as shown in Figure (1). Figure (1) shows the behavior of the considered system without input for the initial condition $x_0 = [1 \quad 2]^T$. Our SOS approach based on the polynomial framework can find fourth order polynomial Lyapunov function satisfying conditions (7), (8), (9) and (10) under $\varepsilon_i(x) \geq 0$, $\forall x \in \Re$, $(i = 1, \ldots, 4)$.

For $\tau = 1$, polynomial matrices computed using Theorem 1 are given by :

$$P(x) = \begin{bmatrix} 139.82x_1^2 + 51.39x_1x_2 + 38.122x_2^2 & -57.907x_1^2 - 30.959x_1x_2 - 19.534x_2^2 \\ -57.907x_1^2 - 30.959x_1x_2 - 19.534x_2^2 & 302.93x_1^2 + 56.664x_1x_2 + 108.43x_2^2 \end{bmatrix} \quad (24)$$

$$Z(x) = \begin{bmatrix} 39.23x_1^2 + 7.8168x_1x_2 + 33.286x_2^2 & 1.0206x_1^2 + 2.4182x_1x_2 - 0.11211x_2^2 \\ 1.0206x_1^2 + 2.4182x_1x_2 - 0.11211x_2^2 & 4.5231x_1^2 + 1.4752x_1x_2 + 0.85523x_2^2 \end{bmatrix} \quad (25)$$

$$S(x) = \begin{bmatrix} 93.872x_1^2+12.187x_1x_2+23.265x_2^2 & 13.32x_1^2-12.373x_1x_2+3.8793x_2^2 \\ 13.32x_1^2-12.373x_1x_2+3.8793x_2^2 & 63.287x_1^2+14.366x_1x_2+46.125x_2^2 \end{bmatrix} \quad (26)$$

To the best of our knowledge, this paper represents the first attempt to apply SOS techniques to polynomial fuzzy systems using such Lyapunov-Krasovskii Functional. The delay-dependent SOS stability conditions scheduled in this article are more relaxed than those cited in [8].Therefore, it appears from this example that our result improves the existing ones.

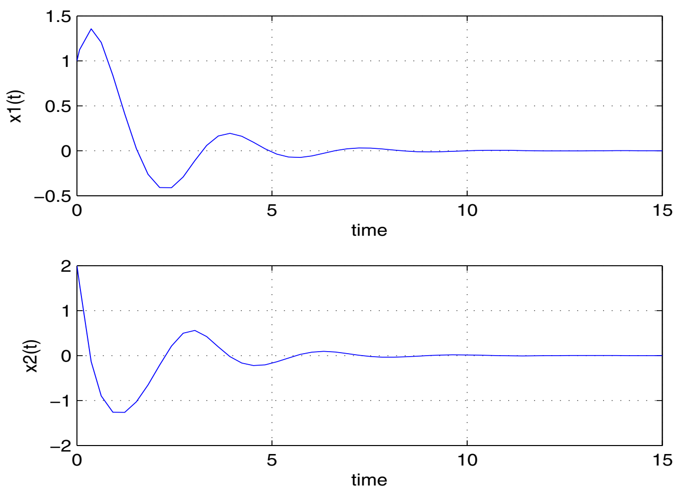

FIGURE 1. State response

V. CONCLUSION

A sum of squares (SOS) approach to stability analysis of polynomial fuzzy systems with time-delay has been presented. First, an overview about polynomial fuzzy models with time-delay has been presented. This class of systems is more general and more efficient than the well-known T-S fuzzy model. Secondly, stability conditions of polynomial fuzzy systems have been achieved based on polynomial Lyapunov-Krasovskii Functional. The stability conditions proposed in this paper are given in terms of SOS which are more general than the existing LMI-based approaches for T-S fuzzy model. The pertinent feature of the established stability conditions is that they can be numerically solved via the recently developed SOSTOOLS. A numerical example has been introduced to validate the effectiveness of the proposed approach.

RÉFÉRENCES

[1] B. Chen and X. Liu. Delay-dependent robust h-infinity control for t-s fuzzy systems with time delay. *IEEE Transactions on Fuzzy Systems*, 13 :238–249, 2005.

[2] Proceedings Asian Control Conference, editor. *Nonlinearcontrol synthesis by sum of squares optimization : A Lyapunov-based approach*, Feb. 2004.

[3] H. Gassara, A. El Hajjaji, M. Kchaou, and M. Chaabane. Robust h-infinity reliable control of time delay nonlinear systems via takagi-sugeno fuzzy models. *International Journal of Systems Science*, iFirst :1–15, 2012.

[4] K. Guelton, N. Manamanni, D. L. Koumba-Emianiwe, and C. D. Chinh. Sos stability conditions for nonlinear systems based on a polynomial fuzzy lyapunov function. Milano(Italy), August 28 - September 2, 2011 2011.

[5] T. Guerra, A. Kruszewski, L. Vermeiren, and H. Tirmant. Conditions of output stabilization for nonlinear models in the takagi-sugeno's form. *Fuzzy Sets and Systems*, 157 :1248 – 1259, 2006.

[6] W. Li and W. Wang. Guaranteed cost control for polynomial discrete fuzzy time delay systems by sum-of-squares approach. 2012.

[7] NAFIPS, editor. *Stability analysis of fuzzy systems using Lyapunov's direct method*, Canada, June 1999.

[8] Proceedings of the 10th World Congress on Intelligent Control and Automation, editors. *Guaranteed Cost Control for Polynomial Fuzzy Time Delay Systems by Sum-of-Squares Approach*, July 6-8 2012.

[9] M. Oudghiri, M. Chadli, and A. ElHajjaji. One-step procedure for robust output fuzzy control. pages pp. 1 –6, Athens, Greece, 2007.

[10] P.A. Parrilo. *Structured semidefinite programs and semialgebraic geometly methods in robustness and optimization*. PhD thesis, California Institute Technology, Pasadena, 2000.

[11] S. Prajna, A. Papachristodoulou, P. Seiler, and P.A. Parrilo. *SOSTOOLS : Sum of Squares Optimization Toolbox for MATLAB, Version 2.00*. California Inst. Technol., Pasadena, 2004.

[12] S. Prajna, A. Papachristodoulou, P. Seiler, P.A. Parrilo, J. Anderson1, and G. Valmorbida1. *SOSTOOLS : Sum of Squares Optimization Toolbox for MATLAB, Version 3.00*, 3rd October 2013.

[13] T. Takagi and M. Sugeno. Fuzzy identification of systems and its applications to modeling and control. *IEEE Transactions Systems*, SMC-15(1) :116–132, Jan/Feb 1985.

[14] K. Tanaka, K. Yamauchi, H. Ohtake, and H. Wang. Guaranteed cost control of polynomial fuzzy systems via a sum-of-squares. New Orleans, LA, USA, 12-14 December 2007.

[15] K. Tanaka, H. Yosihida, H. Ohtake, and H. Wang. Stabilization of polynomial fuzzy systems via a sum of squares approach. Singapore, 1-3 October 2007 2007.

[16] K. Tanaka, H. Yosihida, H. Ohtake, and H. Wang. A sum-of-squares approach to stability analysis of polynomial fuzzy systems. New York City, USA, 11-13 July 2007.

[17] K. Tanaka, H. Yosihida, H. Ohtake, and H. Wang. A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems. In IEEE TRANSACTIONS ON FUZZY SYSTEMS, editor, *A Sum-of-Squares Approach to Modeling and Control of Nonlinear Dynamical Systems With Polynomial Fuzzy Systems*, volume 17, August 2009 2009.

[18] Y. Wang, L. Xie, and Souza. Robust control of a class of uncertain nonlinear systems. *Systems control letters*, 19 :139 – 149, 1992.

[19] F. Zhang, L. Li, and W. Wang. Stability and stabilization for a class of polynomial discrete fuzzy systems with time delay by sum-of-squares optimization. 2011.