Output-Feedback Tracking Control for Polynomial Fuzzy-Model-Based Control Systems

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***Abstract*—This paper presents the output-feedback tracking control of the polynomial fuzzy-model-based control system which consists of a polynomial fuzzy model representing the nonlinear plant and an output-feedback polynomial fuzzy controller connected in a closed loop. The output-feedback polynomial fuzzy controller is employed to drive the system states of the nonlinear plant to follow those of a stable reference model subject to an H_∞ performance. Based on the Lyapunov stability theory, sum-of-squares-based stability conditions are obtained to determine the system stability and facilitate the control synthesis. A feasible solution can be found numerically using the third-party Matlab toolbox SOSTOOLS. Simulation results are provided to demonstrate the merits of the proposed control approach.**

***Index Terms*—Fuzzy tracking control, output feedback, polynomial fuzzy systems, stability analysis, sum of squares.**

NONLINEAR control is a challenging task due to the system nonlinearity is not easily handled in a systematic way. Many control design methods have been proposed for nonlinear systems, such as fuzzy control [1]–[11] and neural network control. Fuzzy-model-based (FMB) control offers an effectively approach to control the nonlinear plant. With the T-S fuzzy model [12], [13], a nonlinear plant can be represented in a general from as a weighted sum of linear sub-systems which locally describe the system dynamics [14]–[16]. By the FMB approach, the fault-tolerant control problems have been investigated in [8], [9]. Based on the local linear sub-system, a linear sub-controller can be designed. A set of fuzzy rules is then employed to combine these linear sub-controllers to form a fuzzy controller [17]. An FMB control system is formed by connecting the nonlinear plant represented by the T-S fuzzy model and a fuzzy controller connected in a closed loop. Other control design results for the T-S fuzzy systems were developed in [18]–[26].

System stability is one of the essential issues considered in the FMB control problems. Lyapunov-based approach is the most popular one to investigate the system stability of the FMB control systems. In [17], the parallel distributed compensation (PDC) design concept was proposed to design the fuzzy controller. Based on the Lyapunov stability theory, the FMB control system is guaranteed to be asymptotically stable if there exists a common solution to a set of linear matrix inequalities (LMIs) [27]. A feasible solution to the LMIs can be found numerically using convex programming techniques. By introducing some slack matrix variables [28], the stability conditions can be relaxed subject to the PDC design approach. Further relaxation can be achieved with the introduction of more slack variables in different ways [29], [30]. However, computational demand for finding the solution will increase.

Sum-of-squares (SOS) decomposition for multivariate polynomials plays an important role to determine positivity of a polynomial function [31]. If a polynomial function can be represented in a form of SOS, the polynomial function can be shown to be positive. This concept is widely used in the stability analysis using the Lyapunov stability theory. Based on the SOS decomposition techniques, construction of polynomial Lyapunov function was formulated as semi-definite programming [32], which is shown to be an effective technique investigating the stability analysis and control synthesis problems for control systems.

Recently, the T-S fuzzy model has been extended to polynomial fuzzy model which allows polynomial variables appearing in the sub-systems to represent a wider class of nonlinear plants. Because of the existence of the polynomial variables, the LMI-based approach is not suitable for stability analysis as the existing LMI solvers cannot deal with polynomial variables directly in the LMI-based stability conditions. Instead, SOS-based analysis approach [32] can be employed. SOS-based stability conditions were obtained to guarantee the the stability of the polynomial FMB (PFMB) control systems [33]–[35]. More relaxed SOS-based stability conditions can be found in [36]. A feasible solution to the SOS-based stability conditions can be found numerically using the third-party Matlab toolbox SOSTOOLS [37].

In [33]–[36, 38], [39], only the stabilization control problem was considered that the system states of the nonlinear plant are driven to the origin. Compared to the stabilization control problem, the tracking control problem [40] is more challenging that a controller is employed to drive the system states of the nonlinear plant to follow a reference or the system states of a stable reference model. Adaptive and/or observer-based fuzzy tracking control were proposed in [41]–[44]. With the adaptive rules, the parameters of the controller are updated to cope with the parameter uncertainties. Compared with the fuzzy controller without adaptive capability, the adaptive fuzzy controller exhibits enhanced robustness property. With the fuzzy observer, the system output can be used to estimate the system states

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for feedback compensation. Consequently, the tracking control strategy can be extended to a wider class of nonlinear systems.

In this paper, the output-feedback tracking control problem for PFMB control systems is considered under the SOS-based framework. Unlike the full-state feedback control approach [40], [43], [45], [46], tracking control is realized using the system output only. The system stability is investigated based on the Lyapunov stability theory under the SOS-based analysis approach. SOS-based stability conditions are derived to guarantee the system stability subject to an H_∞ performance and facilitate the controller synthesis. In [33]–[35], the membership functions were not considered in the stability analysis which lead to conservative stability conditions. It was shown in our preliminary result [36], [38], [39], [47] that the SOS-based stability conditions including the membership functions are able to relax the stability analysis result. In this paper, to obtain relaxed stability conditions, the membership functions are considered in the stability analysis.

The organization of this paper is as follows. In Section I, the PFMB system, stable reference model, and output-feedback polynomial fuzzy controller are introduced. In Section II, the stability analysis is carried out based on the Lyapunov stability theory. SOS-based stability conditions are derived to guarantee the system stability of the PFMB control systems and aid the design of the output-feedback polynomial fuzzy controllers. In Section III, simulation examples are given to show the merits of the proposed approach. In Section IV, a conclusion is drawn.

I. Polynomial Fuzzy Model, Reference Model, and Polynomial Fuzzy Controller

Throughout the paper, the following notations are adopted [32]. The monomial in $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is defined as $x_1^{d_1}(t)x_2^{d_2}(t)\ldots x_n^{d_n}(t)$, d_i is a nonnegative integer, $i = 1, 2, \ldots, n$. The degree of a monomial is defined as $d = \sum_{i=1}^{n} d_i$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is defined as a finite linear combination of monomials with real coefficients. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is a sum of squares if it can be written as $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^{m} \mathbf{q}_j(\mathbf{x}(t))^2$ where $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial and m is a non-zero integer. Hence, it can be seen that $\mathbf{p}(\mathbf{x}(t)) \geq 0$ if it is an SOS. It is stated in [48] that the polynomial $\mathbf{p}(\mathbf{x}(t))$ being an SOS can be represented in the form of $\hat{\mathbf{x}}(t)^T\mathbf{Q}\hat{\mathbf{x}}(t)$ where $\mathbf{Q}$ is a positive semi-definite matrix. The problem of finding a $\mathbf{Q}$ can be formulated as a semi-definite program. SOSTOOLS is a third-party Matlab toolbox to find numerically the matrix $\mathbf{Q}$ to solve the solution to SOS programs. The technical details of SOSTOOLS can be found in [31]. The expressions of $\mathbf{M} > 0$, $\mathbf{M} \geq 0$, $\mathbf{M} < 0$ and $\mathbf{M} \leq 0$ denote a positive, semi-positive, negative, semi-negative definite matrix $\mathbf{M}$, respectively.

In this section, we consider a PFMB control system consisting of a nonlinear plant represented by a polynomial fuzzy model [33], [34] and an output-feedback polynomial fuzzy controller connected in a closed loop. The output-feedback polynomial fuzzy controller is employed to drive the system states of the nonlinear plant to follow those of a stable reference model. Based on the Lyapunov stability theory, the system stability of the PFMB control systems is investigated using the SOS-based approach.

A. Polynomial Fuzzy Model

Let p be the number of fuzzy rules describing the behavior of the nonlinear plant. The i-th rule is of the following format:

$$\text{Rule } i\text{: IF } f_1(\mathbf{y}(t)) \text{ is } M_1^i \text{ AND} \ldots \text{AND } f_\Psi(\mathbf{y}(t)) \text{ is } M_\Psi^i$$
$$\text{THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\,\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\,\mathbf{u}(t) \quad (1)$$

where M_α^i is a fuzzy term of rule i corresponding to the function $f_\alpha(\mathbf{x}(t))$, $\alpha = 1, 2, \ldots, \Psi$; $i = 1, 2, \ldots, p$; Ψ is a positive integer; $\mathbf{x}(t) \in \Re^n$ is the system state vector; $\mathbf{y}(t) \in \Re^l$ is the output vector; $\mathbf{A}_i(\mathbf{x}(t)) \in \Re^{n\times N}$ and $\mathbf{B}_i(\mathbf{x}(t)) \in \Re^{n\times m}$ are the known polynomial system and input matrices, respectively; $\hat{\mathbf{x}}(\mathbf{x}(t)) \in \Re^N$ is a vector of monomials in $\mathbf{x}(t)$; $\mathbf{u}(t) \in \Re^m$ is the input vector. It is assumed that $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{0}$ if and only if $\mathbf{x}(t) = \mathbf{0}$. The system dynamics and output are defined as

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{y}(t))\left(\mathbf{A}_i(\mathbf{x}(t))\,\hat{\mathbf{x}}(\mathbf{x}(t)) + \mathbf{B}_i(\mathbf{x}(t))\,\mathbf{u}(t)\right) \quad (2)$$

$$\mathbf{y}(t) = \mathbf{C}\hat{\mathbf{x}}(\mathbf{x}(t)) \quad (3)$$

where $\mathbf{C} \in \Re^{l\times N}$ is a constant output matrix

$$\sum_{i=1}^{p} w_i(\mathbf{y}(t)) = 1, w_i(\mathbf{y}(t)) \geq 1 \forall i \quad (4)$$

$$w_i(\mathbf{y}(t)) = \frac{\prod_{l=1}^{\Psi} \mu_{M_l^i}(f_l(\mathbf{y}(t)))}{\sum_{k=1}^{p}\prod_{l=1}^{\Psi} \mu_{M_l^k}(f_l(\mathbf{y}(t)))} \forall i \quad (5)$$

$w_i(\mathbf{y}(t))$ is the normalized grade of membership, $\mu_{M_\alpha^i}(f_\alpha(\mathbf{y}(t)))$, $\alpha = 1, 2, \ldots, \Psi$, is the grade of membership corresponding to the fuzzy term of M_α^i.

Remark 1: The polynomial fuzzy model (2) is reduced to the traditional T-S fuzzy model [28] when $\mathbf{A}_i(\mathbf{x}(t))$ and $\mathbf{B}_i(\mathbf{x}(t))$ are constant matrices for all i and $\hat{\mathbf{x}}(\mathbf{x}(t)) = \mathbf{x}(t)$.

B. Reference Model

A stable reference model is defined as follows:

$$\dot{\mathbf{x}}_\mathbf{r}(t) = \mathbf{A}_\mathbf{r}\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x}_\mathbf{r}(t)) + \mathbf{B}_\mathbf{r}\mathbf{r}(t) \quad (6)$$

$$\mathbf{y}_\mathbf{r}(t) = \mathbf{C}\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x}_\mathbf{r}(t)) \quad (7)$$

where $\mathbf{x}_\mathbf{r}(t) \in \Re^N$ is the state vector of the reference model, $\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x}_\mathbf{r}(t)) \in \Re^N$ is a vector of monomials in $\mathbf{x}_\mathbf{r}(t)$ as the entries, $\mathbf{A}_\mathbf{r} \in \Re^{n\times N}$ and $\mathbf{B}_\mathbf{r} \in \Re^{n\times m}$ are the constant system and input matrices, respectively, $\mathbf{r}(t) \in \Re^m$ is the reference input vector, $\mathbf{y}_\mathbf{r}(t) \in \Re^l$ is the output vector of the reference model.

Remark 2: The system and input matrices $\mathbf{A}_\mathbf{r}$ and $\mathbf{B}_\mathbf{r}$ are not limited to be constant matrices. If they are function of the system states, we have polynomial matrices $\mathbf{A}_\mathbf{r}(\mathbf{x}_\mathbf{r}(t))$ and $\mathbf{B}_\mathbf{r}(\mathbf{x}_\mathbf{r}(t))$. However, it is required to make sure that the reference model is stable.

C. Output-Feedback Polynomial Fuzzy Controller

An output-feedback polynomial fuzzy controller is proposed to drive the system states of the nonlinear plant in the form of (2) to follow those of the stable reference model (6). Define the state error as

$$\hat{\mathbf{e}}(t) = \hat{\mathbf{x}}\left(\mathbf{x}(t)\right) - \hat{\mathbf{x}}_{\mathbf{r}}\left(\mathbf{x}_{\mathbf{r}}(t)\right). \quad (8)$$

From (3), (7) and (8), the output error is defined as follows:

$$\mathbf{e}_{\mathbf{y}}(t) = \mathbf{y}(t) - \mathbf{y}_{\mathbf{r}}(t) = \mathbf{C}\hat{\mathbf{e}}(t). \quad (9)$$

Define $\mathbf{h}(t) = (\mathbf{y}(t), \mathbf{y}_{\mathbf{r}}(t))$. The output-feedback polynomial fuzzy controller is described by the following p rules:

$$\text{Rule } j: \text{IF } f_1(\mathbf{y}(t)) \text{ is } M_1^j \text{ AND} \ldots \text{AND } f_\Psi(\mathbf{y}(t)) \text{ is } M_\Psi^j$$
$$\text{THEN } \mathbf{u}(t) = \mathbf{F}_j\left(\mathbf{h}(t)\right)\mathbf{e}_{\mathbf{y}}(t) + \mathbf{G}_j\left(\mathbf{h}(t)\right)\mathbf{y}_{\mathbf{r}}(t) \quad (10)$$

where $\mathbf{F}_j(\mathbf{h}(t)) \in \Re^{m\times N}$ and $\mathbf{G}_j(\mathbf{h}(t)) \in \Re^{m\times N}$, $j = 1, 2, \ldots, p$, are the polynomial feedback gains to be determined. The output-feedback polynomial fuzzy controller is defined as

$$\mathbf{u}(t) = \sum_{j=1}^{p} w_j\left(\mathbf{y}(t)\right)\left(\mathbf{F}_j\left(\mathbf{h}(t)\right)\mathbf{C}\hat{\mathbf{e}}(t) + \mathbf{G}_j\left(\mathbf{h}(t)\right)\mathbf{C}\hat{\mathbf{x}}_{\mathbf{r}}\left(\mathbf{x}_{\mathbf{r}}(t)\right)\right). \quad (11)$$

Remark 3: The feedback gains $\mathbf{F}_j(\mathbf{h}(t))$ and $\mathbf{G}_j(\mathbf{h}(t))$ are reduced to scalar matrices. That is, $\mathbf{F}_j$ and $\mathbf{G}_j$, when the degree of polynomial is chosen to be 0.

Remark 4: The polynomial fuzzy controller (11) is an output-feedback polynomial fuzzy controller and becomes a full state-feedback polynomial fuzzy controller when $\mathbf{C}$ is a full rank matrix, for example, $\mathbf{C} = \mathbf{I}$.

II. Stability Analysis

For brevity, in the following analysis, $w_i(\mathbf{y}(t))$ is denoted as w_i and the time t associated with the variables is dropped for the situation without ambiguity, e.g., $\mathbf{h}(t)$, $\mathbf{x}(t)$, $\hat{\mathbf{x}}(\mathbf{x}(t))$, $\mathbf{x}_{\mathbf{r}}(t)$, $\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}(t))$, $\mathbf{y}(t)$, $\hat{\mathbf{y}}_{\mathbf{r}}(t)$, $\hat{\mathbf{e}}(t)$, $\mathbf{e}_{\mathbf{y}}(t)$ and $\mathbf{r}(t)$ are denoted as $\mathbf{h}$, $\mathbf{x}$, $\hat{\mathbf{x}}(\mathbf{x})$, $\mathbf{x}_{\mathbf{r}}$, $\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}})$, $\mathbf{y}$, $\hat{\mathbf{y}}_{\mathbf{r}}$, $\hat{\mathbf{e}}$, $\mathbf{e}_{\mathbf{y}}$ and $\mathbf{r}$, respectively.

Considering the polynomial fuzzy model (2) and the output-feedback polynomial fuzzy controller (11), with the property of the membership functions (4), i.e., $\sum_{i=1}^{p} w_i = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j = 1$, we have

$$\dot{\mathbf{x}} = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\left(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{F}_j(\mathbf{h})\mathbf{C}\right)\hat{\mathbf{e}} + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\left(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{G}_j(\mathbf{h})\mathbf{C}\right)\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}). \quad (12)$$

Denote $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ and $\hat{\mathbf{x}}(\mathbf{x}) = [\hat{x}_1(\mathbf{x}), \hat{x}_2(\mathbf{x}), \ldots, \hat{x}_N(\mathbf{x})]^T$. From (12), we have

$$\dot{\hat{\mathbf{x}}}(\mathbf{x}) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\left(\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{F}_j(\mathbf{h})\mathbf{C}\right)\hat{\mathbf{e}} + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\left(\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{h})\mathbf{C}\right)\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}) \quad (13)$$

where $\tilde{\mathbf{A}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{A}_i(\mathbf{x})$, $\tilde{\mathbf{B}}_i(\mathbf{x}) = \mathbf{T}(\mathbf{x})\mathbf{B}_i(\mathbf{x})$ and $\mathbf{T}(\mathbf{x}) \in \Re^{N\times n}$ with its $\alpha\beta$ element defined as

$$T_{\alpha\beta}(\mathbf{x}) = \frac{\partial \hat{x}_\alpha(\mathbf{x})}{\partial x_\beta}, \alpha = 1, 2, \ldots, N; \beta = 1, 2, \ldots, n. \quad (14)$$

Similarly, denote $\mathbf{x}_{\mathbf{r}} = [x_{r_1}, x_{r_2}, \ldots, x_{r_n}]^T$ and $\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}) = [\hat{x}_{r_1}(\mathbf{x}_{\mathbf{r}}), \hat{x}_{r_2}(\mathbf{x}_{\mathbf{r}}), \ldots, \hat{x}_{r_N}(\mathbf{x}_{\mathbf{r}})]^T$. From (6), we have

$$\dot{\hat{\mathbf{x}}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}) = \frac{\partial \hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}})}{\partial \mathbf{x}_{\mathbf{r}}}\frac{d\mathbf{x}_{\mathbf{r}}}{dt} = \mathbf{H}(\mathbf{x}_{\mathbf{r}})\dot{\mathbf{x}}_{\mathbf{r}} = \tilde{\mathbf{A}}_{\mathbf{r}}\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}) + \tilde{\mathbf{B}}_{\mathbf{r}}\mathbf{r} \quad (15)$$

where $\tilde{\mathbf{A}}_{\mathbf{r}} = \mathbf{H}(\mathbf{x}_{\mathbf{r}})\mathbf{A}_{\mathbf{r}}$, $\tilde{\mathbf{B}}_{\mathbf{r}} = \mathbf{H}(\mathbf{x}_{\mathbf{r}})\mathbf{B}_{\mathbf{r}}$ and $\mathbf{H}(\mathbf{x}_{\mathbf{r}}) \in \Re^{N\times n}$ with its $\alpha\beta$ element defined as

$$H_{\alpha\beta}(\mathbf{x}_{\mathbf{r}}) = \frac{\partial \hat{x}_{r_\alpha}(\mathbf{x}_{\mathbf{r}})}{\partial x_{r_\beta}}, \alpha = 1, 2, \ldots, N; \beta = 1, 2, \ldots, n. \quad (16)$$

Considering the state error vector (8), from (13) and (15), we have

$$\begin{aligned}\dot{\hat{\mathbf{e}}} &= \dot{\hat{\mathbf{x}}}(\mathbf{x}) - \dot{\hat{\mathbf{x}}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\left(\tilde{\mathbf{A}}_i(\mathbf{x}) + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{F}_j(\mathbf{h})\mathbf{C}\right)\hat{\mathbf{e}} \\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\left(\tilde{\mathbf{A}}_i(\mathbf{x}) - \tilde{\mathbf{A}}_{\mathbf{r}} + \tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{h})\mathbf{C}\right)\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}) - \tilde{\mathbf{B}}_{\mathbf{r}}\mathbf{r}. \quad (17)\end{aligned}$$

Define

$$\mathbf{\Gamma} = \left[\mathbf{C}^T\left(\mathbf{C}\mathbf{C}^T\right)^{-1} \quad ortc(\mathbf{C}^T)\right] \quad (18)$$

where $\mathbf{\Gamma} \in \Re^{N\times N}$ and $ortc(\mathbf{C}^T)$ denotes the orthogonal complement of $\mathbf{C}^T$[49]. It is assumed that $\mathbf{\Gamma}^{-1}$ exists in the following analysis. From (18), we have

$$\mathbf{C}\mathbf{\Gamma} = [\mathbf{I}_l \; \mathbf{0}] \quad (19)$$

where $\mathbf{I}_l \in \Re^{l\times l}$ is the identity matrix.

Remark 5: To facilitate the stability analysis [32], [33], the row indices that the entries of the entire row of $\mathbf{B}_i(\mathbf{x})$ for all i are all zeros are denoted by $\mathbf{J} = \{j_1, j_2, \ldots, j_q\}$. Similarly, the row indices that the entries of the entire row of $\mathbf{B}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}})$ are all zeros are denoted by $\mathbf{K} = \{k_1, k_2, \ldots, k_s\}$.

Define $0 < \mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N\times N}$, $\tilde{\mathbf{x}} = (x_{j_1}, x_{j_2}, \ldots, x_{j_q}, x_{r_{k_1}}, x_{r_{k_2}}, \ldots, x_{r_{k_s}})$. Inspired by [49], we choose

$$\mathbf{X}(\tilde{\mathbf{x}}) = \begin{bmatrix}\mathbf{X}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_{22}(\tilde{\mathbf{x}})\end{bmatrix} \quad (20)$$

where $\mathbf{X}_{11} \in \Re^{l\times l}$, $\mathbf{X}_{22}(\tilde{\mathbf{x}}) \in \Re^{(N-l)\times(N-l)}$.

Remark 6: Assuming that $\mathbf{X}(\tilde{\mathbf{x}}) > 0$ (a nonsingular matrix), it can be obtained that $\mathbf{X}(\tilde{\mathbf{x}})^{-1} = \text{diag}\{\mathbf{X}_{11}^{-1}, \mathbf{X}_{22}(\tilde{\mathbf{x}})^{-1}\}$ because of the inversion of a blockwise matrix. As a result, if $\mathbf{X}(\tilde{\mathbf{x}}) > 0$, both $\mathbf{X}_{11}^{-1}$ and $\mathbf{X}_{22}^{-1}(\tilde{\mathbf{x}})$ exist.

The polynomial feedback gains are defined as

$$\mathbf{F}_j(\mathbf{h}) = \mathbf{M}_j(\mathbf{h})\mathbf{X}_{11}^{-1}, \mathbf{G}_j(\mathbf{h}) = \mathbf{N}_j(\mathbf{h})\mathbf{X}_{11}^{-1} \quad (21)$$

where $\mathbf{M}_j(\mathbf{h}) \in \Re^{m\times l}$ and $\mathbf{N}_j(\mathbf{h}) \in \Re^{m\times l}$.

Denoting $\hat{\mathbf{v}} = \mathbf{\Gamma}^{-1}\hat{\mathbf{e}}$, (17) can be written as follows:

$$\begin{aligned}\dot{\hat{\mathbf{v}}} =& \mathbf{\Gamma}^{-1}\dot{\hat{\mathbf{e}}} \\ =& \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\Big(\mathbf{\Gamma}^{-1}\tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) \\ & + \mathbf{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{F}_j(\mathbf{h})\mathbf{C}\mathbf{\Gamma}\mathbf{X}(\tilde{\mathbf{x}})\Big)\mathbf{X}(\tilde{\mathbf{x}})^{-1}\mathbf{\Gamma}^{-1}\hat{\mathbf{e}} \\ & + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\Big(\mathbf{\Gamma}^{-1}\left(\tilde{\mathbf{A}}_i(\mathbf{x}) - \tilde{\mathbf{A}}_{\mathbf{r}}\right)\mathbf{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) \\ & + \mathbf{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x})\mathbf{G}_j(\mathbf{h})\mathbf{C}\mathbf{\Gamma}\mathbf{X}(\tilde{\mathbf{x}})\Big) \\ & \times \mathbf{X}(\tilde{\mathbf{x}})^{-1}\mathbf{\Gamma}^{-1}\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}) - \mathbf{\Gamma}^{-1}\tilde{\mathbf{B}}_{\mathbf{r}}\mathbf{r}. \end{aligned} \quad (22)$$

Referring to (22), with (18)–(21), we have

$$\mathbf{F}_j(\mathbf{h})\mathbf{C}\mathbf{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) = [\mathbf{M}_j(\mathbf{h}) \quad \mathbf{0}] \quad (23)$$

$$\mathbf{G}_j(\mathbf{h})\mathbf{C}\mathbf{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) = [\mathbf{N}_j(\mathbf{h}) \quad \mathbf{0}]. \quad (24)$$

With (23) and (24), (22) becomes

$$\begin{aligned}\dot{\hat{\mathbf{v}}} =& \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\Big(\mathbf{\Gamma}^{-1}\tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x}) \\ & \times [\mathbf{M}_j(\mathbf{h}) \quad \mathbf{0}]\Big)\mathbf{X}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{v}} \\ & + \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\Big(\mathbf{\Gamma}^{-1}\left(\tilde{\mathbf{A}}_i(\mathbf{x}) - \tilde{\mathbf{A}}_{\mathbf{r}}\right)\mathbf{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) \\ & + \mathbf{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x})[\mathbf{N}_j(\mathbf{h}) \quad \mathbf{0}]\Big) \\ & \times \mathbf{X}(\tilde{\mathbf{x}})^{-1}\mathbf{\Gamma}^{-1}\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}) - \mathbf{\Gamma}^{-1}\tilde{\mathbf{B}}_{\mathbf{r}}\mathbf{r} \\ =& \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{\Phi}_{ij}(\mathbf{x}, \mathbf{x}_{\mathbf{r}})\mathbf{z} \end{aligned} \quad (25)$$

where $\mathbf{\Phi}_{ij}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) = [\mathbf{\Phi}_{ij}^{(1)}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) \quad \mathbf{\Phi}_{ij}^{(2)}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) \quad \mathbf{\Phi}_{ij}^{(3)}(\mathbf{x}, \mathbf{x}_{\mathbf{r}})]$, $\mathbf{\Phi}_{ij}^{(1)}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) = \mathbf{\Gamma}^{-1}\tilde{\mathbf{A}}_i(\mathbf{x})\mathbf{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x})[\mathbf{M}_j(\mathbf{h}) \quad \mathbf{0}]$, $\mathbf{\Phi}_{ij}^{(2)}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) = \mathbf{\Gamma}^{-1}(\tilde{\mathbf{A}}_i(\mathbf{x}) - \tilde{\mathbf{A}}_{\mathbf{r}})\mathbf{\Gamma}\mathbf{X}(\tilde{\mathbf{x}}) + \mathbf{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x})[\mathbf{N}_j(\mathbf{h}) \quad \mathbf{0}]$, $\mathbf{\Phi}_{ij}^{(3)}(\mathbf{x}, \mathbf{x}_{\mathbf{r}}) = -\mathbf{\Gamma}^{-1}\tilde{\mathbf{B}}_{\mathbf{r}}$, $\mathbf{z} = \begin{bmatrix}\mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3\end{bmatrix} = \begin{bmatrix}\mathbf{X}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{v}} \\ \mathbf{X}(\tilde{\mathbf{x}})^{-1}\mathbf{\Gamma}^{-1}\mathbf{x}_{\mathbf{r}} \\ \mathbf{r}\end{bmatrix} = \begin{bmatrix}\mathbf{X}(\tilde{\mathbf{x}})^{-1}\mathbf{\Gamma}^{-1}\hat{\mathbf{e}} \\ \mathbf{X}(\tilde{\mathbf{x}})^{-1}\mathbf{\Gamma}^{-1}\mathbf{x}_{\mathbf{r}} \\ \mathbf{r}\end{bmatrix}$.

Remark 7: It can be seen from (23) and (24) that $\mathbf{F}_j(\mathbf{h})\mathbf{C}\mathbf{\Gamma}\mathbf{X}(\tilde{\mathbf{x}})$ and $\mathbf{G}_j(\mathbf{h})\mathbf{C}\mathbf{\Gamma}\mathbf{X}(\tilde{\mathbf{x}})$ can be reduced to $[\mathbf{M}_j(\mathbf{h})\ \mathbf{0}]$ and $[\mathbf{N}_j(\mathbf{h})\ \mathbf{0}]$, respectively, by choosing $\mathbf{\Gamma}$ defined in (18) leading to property (19), i.e., $\mathbf{C}\mathbf{\Gamma} = [\mathbf{I}_l\ \mathbf{0}]$ and $\mathbf{X}(\tilde{\mathbf{x}})$ defined in (20). As a result, the terms $\mathbf{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x})[\mathbf{M}_j(\mathbf{h})\ \mathbf{0}]$ and $\mathbf{\Gamma}^{-1}\tilde{\mathbf{B}}_i(\mathbf{x})[\mathbf{N}_j(\mathbf{h})\ \mathbf{0}]$ in $\mathbf{\Phi}_{ij}^{(1)}(\mathbf{x}, \mathbf{x}_{\mathbf{r}})$ and $\mathbf{\Phi}_{ij}^{(2)}(\mathbf{x}, \mathbf{x}_{\mathbf{r}})$, respectively, depend only on $\mathbf{N}_j(\mathbf{h})$ or $\mathbf{M}_j(\mathbf{h})$ such that convex programming techniques can be employed to find their values numerically.

In the following, the system stability of the error system of (25) is investigated. Consider the following polynomial Lyapunov function candidate:

$$V(t) = \hat{\mathbf{v}}^T\mathbf{X}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{v}}. \quad (26)$$

From (26), we have

$$\dot{V}(t) = \dot{\hat{\mathbf{v}}}^T\mathbf{X}(\tilde{\mathbf{x}})^{-1}\hat{\mathbf{v}} + \hat{\mathbf{v}}^T\mathbf{X}(\tilde{\mathbf{x}})^{-1}\dot{\hat{\mathbf{v}}} + \hat{\mathbf{v}}^T\frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt}\hat{\mathbf{v}}. \quad (27)$$

To deal with the term $d\mathbf{X}(\tilde{\mathbf{x}})^{-1}/dt$ in (27) on the right hand side, the following Lemma [32], [33], is introduced.

Lemma 1: For any invertible polynomial matrix $\mathbf{X}(\tilde{\mathbf{x}})$, the following equality holds:

$$\frac{\partial\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k} = -\mathbf{X}(\tilde{\mathbf{x}})^{-1}\frac{\partial\mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}\mathbf{X}(\tilde{\mathbf{x}})^{-1}.$$

Remark 8: Denote $\mathbf{A}_i^{(k)}(\mathbf{x}) \in \Re^N$, $\mathbf{B}_i^{(k)}(\mathbf{x}) \in \Re^m$, $\mathbf{A}_{\mathbf{r}}^{(k)} \in \Re^N$ and $\mathbf{B}_{\mathbf{r}}^{(k)} \in \Re^m$, $i = 1, 2, \ldots, p$, $k = 1, 2, \ldots, n$, as the kth row of $\mathbf{A}_i(\mathbf{x})$, $\mathbf{B}_i(\mathbf{x})$, $\mathbf{A}_{\mathbf{r}}$ and $\mathbf{B}_{\mathbf{r}}$, respectively. It should be noted that $\mathbf{A}_i^{(k)}(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x})$ and $\mathbf{A}_{\mathbf{r}}^{(k)}(\mathbf{x}_{\mathbf{r}})\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}})$ are scalar polynomials for all k. Furthermore, from (2) and (6), we have $\dot{x}_k = \sum_{i=1}^{p} w_i(\mathbf{A}_i^{(k)}(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x}) + \mathbf{B}_i^{(k)}(\mathbf{x})\mathbf{u})$ and $\dot{x}_{r_k} = \mathbf{A}_{\mathbf{r}}^{(k)}\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}}) + \mathbf{B}_{\mathbf{r}}^{(k)}\mathbf{r}$. From Remark 5, we have $\partial\mathbf{X}(\tilde{\mathbf{x}})^{-1}/\partial x_k = \mathbf{0}$ for $k \notin \mathbf{J}$ and $\partial\mathbf{X}(\tilde{\mathbf{x}})^{-1}/\partial x_{r_k} = \mathbf{0}$ for $k \notin \mathbf{K}$. Consequently, we have $\dot{x}_k = \sum_{i=1}^{p} w_i\mathbf{A}_i^{(k)}(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x})$ for $k \in \mathbf{J}$ and $\dot{x}_{r_k} = \mathbf{A}_r^{(k)}\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}})$ for $k \in \mathbf{K}$ which will be used in the following analysis.

Recalling that $\tilde{\mathbf{x}} = (x_{j_1}, x_{j_2}, \ldots, x_{j_q}, x_{r_{k_1}}, x_{r_{k_2}}, \ldots, x_{r_{k_s}})$ and considering the term of $d\mathbf{X}(\tilde{\mathbf{x}})^{-1}/dt$ in (27), from Lemma 1, Remark 5, and Remark 8, we have

$$\begin{aligned}\frac{d\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{dt} &= \sum_{k=1}^{n}\left(\frac{\partial\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_k}\dot{x}_k + \frac{\partial\mathbf{X}(\tilde{\mathbf{x}})^{-1}}{\partial x_{r_k}}\dot{x}_{r_k}\right) \\ &= -\sum_{k\in\mathbf{J}}\mathbf{X}(\tilde{\mathbf{x}})^{-1}\left(\frac{\partial\mathbf{X}(\tilde{\mathbf{x}})}{\partial x_k}\sum_{i=1}^{p} w_i\mathbf{A}_i^{(k)}(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x})\right)\mathbf{X}(\tilde{\mathbf{x}})^{-1} \\ &\quad - \sum_{k\in\mathbf{K}}\mathbf{X}(\tilde{\mathbf{x}})^{-1}\left(\frac{\partial\mathbf{X}(\tilde{\mathbf{x}})}{\partial x_{r_k}}\mathbf{A}_r^{(k)}\hat{\mathbf{x}}_{\mathbf{r}}(\mathbf{x}_{\mathbf{r}})\right)\mathbf{X}(\tilde{\mathbf{x}})^{-1}. \end{aligned} \quad (28)$$

From (25), (27) and (28), we have

$$\dot{V}(t) = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j\mathbf{z}^T\mathbf{\Xi}_{ij}(\mathbf{x}, \mathbf{x}_{\mathbf{r}})\mathbf{z} - \mathbf{z}_1^T\mathbf{z}_1 + \sigma_1^2\mathbf{z}_2^T\mathbf{z}_2 + \sigma_2^2\mathbf{z}_3^T\mathbf{z}_3 \quad (29)$$

where $\boldsymbol{\Xi}_{ij}(\mathbf{x},\mathbf{x_r}) = \begin{bmatrix} \boldsymbol{\Xi}_{ij}^{(11)}(\mathbf{x},\mathbf{x_r}) & * & * \\ \boldsymbol{\Phi}_{ij}^{(2)}(\mathbf{x},\mathbf{x_r})^T & -\sigma_1^2\mathbf{I} & * \\ \boldsymbol{\Phi}_{ij}^{(3)}(\mathbf{x},\mathbf{x_r})^T & \mathbf{0} & -\sigma_2^2\mathbf{I} \end{bmatrix}$, $\boldsymbol{\Xi}_{ij}^{(11)} = \boldsymbol{\Phi}_{ij}^{(1)}(\mathbf{x},\mathbf{x_r}) + \boldsymbol{\Phi}_{ij}^{(1)}(\mathbf{x},\mathbf{x_r})^T + \mathbf{I} - \sum_{k\in\mathbf{J}}(\partial\mathbf{X}(\tilde{\mathbf{x}})/\partial x_k)\, \mathbf{A}_i^{(k)}(\mathbf{x})\hat{\mathbf{x}}(\mathbf{x}) - \sum_{k\in\mathbf{K}}(\partial\mathbf{X}(\tilde{\mathbf{x}})/\partial x_{r_k})\, \mathbf{A}_r^{(k)}\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x_r})$, "$*$" denotes the transposed element at the corresponding position, σ_1 and σ_2 are scalars to be determined.

Remark 9: Because the reference model (6) is stable, the system state $\mathbf{x_r}$ and the reference input $\mathbf{r}$ are bounded. This property will be employed to construct the following H_∞ performance.

Considering

$$\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \boldsymbol{\Xi}_{ij}(\mathbf{x},\mathbf{x_r}) < 0 \tag{30}$$

from (29), we have

$$\dot{V}(t) \le -\mathbf{z}_1^T\mathbf{z}_1 + \sigma_1^2\mathbf{z}_2^T\mathbf{z}_2 + \sigma_2^2\mathbf{z}_3^T\mathbf{z}_3. \tag{31}$$

Considering the termination time of control t_f[40] and taking integration on both sides of (31) with respect to time t, we obtain the following H_∞ performance:

$$\frac{\int_0^{t_f} \mathbf{z}_1^T\mathbf{z}_1 - V(0)}{\int_0^{t_f}\left(\sigma_1^2\mathbf{z}_2^T\mathbf{z}_2 + \sigma_2^2\mathbf{z}_3^T\mathbf{z}_3\right)dt} \le 1. \tag{32}$$

It can be seen from (32) that the tracking performance can be improved with smaller values of σ_1 and σ_2. In order to guarantee the system stability of (25), the inequality of (30) is needed to be satisfied. The third-party Matlab toolbox SOSTOOLS [37] is employed to find a feasible solution numerically. As the membership functions, w_i, are treated as symbolic variables by SOSTOOLS, it is difficult to set up a set of SOS conditions to specify that the membership functions are positive functions, i.e., $0 \le w_i \le 1$. To circumvent the problem, we denote the membership functions of w_i as $\bar{w}_i^2$, $i = 1, 2, \ldots, p$[35], [47], [50], [51]. Consequently, the information of $w_i \ge 0$ can be carried to the SOSTOOLS by using the symbolic variable of $\bar{w}_i^2$ which is obviously a positive function. The analysis result of the output state-feedback tracking control is summarized in the following theorem.

Theorem 1: The output-feedback polynomial fuzzy controller (11) is able to drive the system states of the nonlinear plant in the form of (2) to follow those of the stable reference model (6) subject to the H_∞ performance (32) if there exists pre-defined SOS scalar polynomial functions $\varepsilon_1(\tilde{\mathbf{x}})$ and $\varepsilon_2(\mathbf{x},\mathbf{x_r},\mathbf{w})$ and decision variables, i.e., polynomial matrices $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N\times N}$ in the form of (20), $\mathbf{M}_j(\mathbf{h}) \in \Re^{m\times N}$, $\mathbf{N}_j(\mathbf{h}) \in \Re^{m\times N}$, $j = 1, 2, \ldots, p$, such that the following SOS conditions are satisfied

$$\boldsymbol{\nu}^T\left(\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}})\right)\boldsymbol{\nu} \text{ is SOS,}$$
$$-\boldsymbol{\rho}^T\left(\sum_{i=1}^{p}\sum_{j=1}^{p}\bar{w}_i^2\bar{w}_j^2\boldsymbol{\Xi}_{ij}(\mathbf{x},\mathbf{x_r}) + \varepsilon_2(\mathbf{x},\mathbf{x_r},\mathbf{w})\right)\boldsymbol{\rho} \text{ is SOS}$$

where $\mathbf{w} = [\bar{w}_1^2, \bar{w}_2^2, \ldots, \bar{w}_p^2]$, $\boldsymbol{\nu} \in \Re^N$ and $\boldsymbol{\rho} \in \Re^{2N+m}$ are arbitrary vectors independent of $\mathbf{x}$ and $\mathbf{x_r}$, σ_1 and σ_2 in $\boldsymbol{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})$ are pre-defined scalars, and the feedback gains are defined in (21).

It should be noted that the computational demand for finding the solution of the SOS-based stability conditions in Theorem 1 will increase when the number of decision variables increases. Consequently, the third-party Matlab toolbox, SOSTOOLS [37], cannot produce result even though the SOS-based stability conditions are feasible. In order to reduce the computational demand for finding the solution, the following Corollary reduced from the SOS-based stability conditions in Theorem 1 can be used.

Corollary 1: The output-feedback polynomial fuzzy controller (11) is able to drive the system states of the nonlinear plant in the form of (2) to follow those of the stable reference model (6) subject to the H_∞ performance (32) if there exists pre-defined SOS scalar polynomial functions $\varepsilon_1(\tilde{\mathbf{x}})$ and $\varepsilon_2(\mathbf{x},\mathbf{x_r})$ and decision variables, i.e., polynomial matrices $\mathbf{X}(\tilde{\mathbf{x}}) = \mathbf{X}(\tilde{\mathbf{x}})^T \in \Re^{N\times N}$ in the form of (20), $\mathbf{M}_j(\mathbf{h}) \in \Re^{m\times N}$, $\mathbf{N}_j(\mathbf{h}) \in \Re^{m\times N}$, $j = 1, 2, \ldots, p$, such that the following SOS conditions are satisfied:

$$\boldsymbol{\nu}^T\left(\mathbf{X}(\tilde{\mathbf{x}}) - \varepsilon_1(\tilde{\mathbf{x}})\right)\boldsymbol{\nu} \text{ is SOS}$$
$$-\boldsymbol{\rho}^T\left(\boldsymbol{\Xi}_{ij}(\mathbf{x},\mathbf{x_r}) + \boldsymbol{\Xi}_{ji}(\mathbf{x},\mathbf{x_r}) + \varepsilon_2(\mathbf{x},\mathbf{x_r})\right)\boldsymbol{\rho} \text{ is SOS}$$
$$\forall j = 1, 2, \ldots, p; i < j$$

where $\boldsymbol{\nu} \in \Re^N$, $\boldsymbol{\rho} \in \Re^{2N+m}$ are arbitrary vectors independent of $\mathbf{x}$ and $\mathbf{x_r}$, σ_1 and σ_2 in $\boldsymbol{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})$ are pre-defined scalars, and the feedback gains are defined in (21).

Corollary 1 can be obtained by considering (29) rewritten as follows:

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{z}^T\boldsymbol{\Xi}_{ij}(\mathbf{x},\mathbf{x_r})\mathbf{z} \\ &\quad - \mathbf{z}_1^T\mathbf{z}_1 + \sigma_1^2\mathbf{z}_2^T\mathbf{z}_2 + \sigma_2^2\mathbf{z}_3^T\mathbf{z}_3 \\ &= \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \mathbf{z}^T\left(\boldsymbol{\Xi}_{ij}(\mathbf{x},\mathbf{x_r}) + \boldsymbol{\Xi}_{ji}(\mathbf{x},\mathbf{x_r}\right)\mathbf{z} \\ &\quad - \mathbf{z}_1^T\mathbf{z}_1 + \sigma_1^2\mathbf{z}_2^T\mathbf{z}_2 + \sigma_2^2\mathbf{z}_3^T\mathbf{z}_3. \end{aligned} \tag{33}$$

It can be seen from (31) that if the second SOS-based condition in Corollary 1 is satisfied, we have the inequality (31) that the PFMB control system satisfies the H_∞ performance (32).

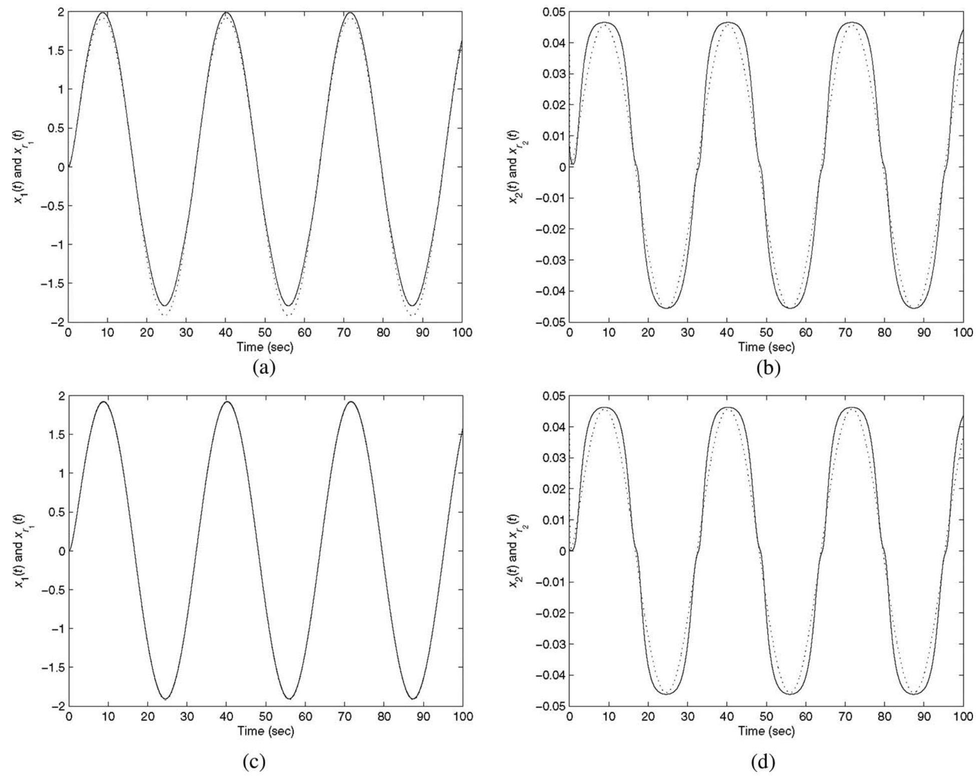


Fig. 1. (a) System responses of $x_1(t)$ (Solid line) and $x_{r_1}(t)$ (Dotted line) with $\sigma_1 = 10$ and $\sigma_2 = 10$. (b) System responses of $x_2(t)$ (Solid line) and $x_{r_2}(t)$ (Dotted line) with $\sigma_1 = 10$ and $\sigma_2 = 10$. (c) System responses of $x_1(t)$ (Solid line) and $x_{r_1}(t)$ (Dotted line) with $\sigma_1 = 7.5$ and $\sigma_2 = 0.1$. (d) System responses of $x_2(t)$ (Solid line) and $x_{r_2}(t)$ (Dotted line) with $\sigma_1 = 7.5$ and $\sigma_2 = 0.1$.

III. Simulation Examples

In this section, three simulation examples are presented to demonstrate the merits of the proposed approach.

A. Simulation Example 1

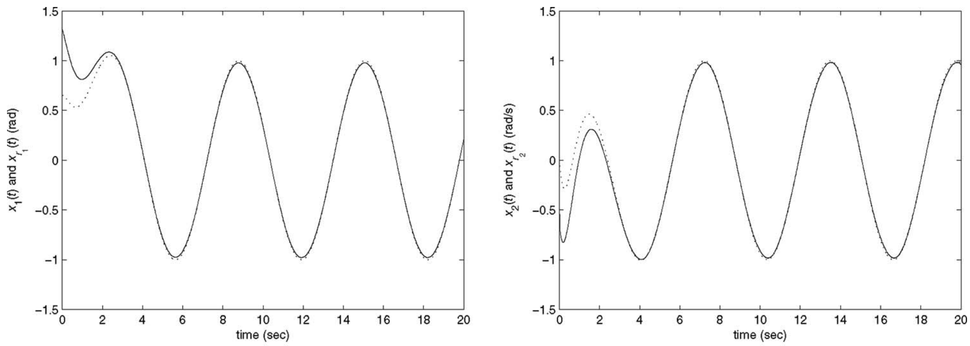
Consider a three-rule polynomial fuzzy model in the form of (2) with the following system, input and output matrices. $\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x}$, $\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x_r}) = \mathbf{x_r}$, $\mathbf{A}_1(x_1) = \begin{bmatrix} 0.59 - 0.12x_1 & -7.29 - 1.82x_1 \\ 0.01 & -2.85 \end{bmatrix}$, $\mathbf{A}_2(x_1) = \begin{bmatrix} 0.02{+}2.25x_1 & -4.64{+}0.72x_1 \\ 0.35 & -8.56, \end{bmatrix}$, $\mathbf{B}_1(x_1) = \begin{bmatrix} 1{+}1.35x_1{+}2.33x_1^2 \\ 0 \end{bmatrix}$, $\mathbf{B}_2(x_1) = \begin{bmatrix} 8{-}0.62x_1{+}0.56x_1^2 \\ 0 \end{bmatrix}$, and $\mathbf{C} = [1\ 0]$.

The membership functions of the polynomial fuzzy model are chosen as $w_1(x_1) = \mu_{M_1^1} = e^{x_1^2/(2\times 0.5^2)}$ and $w_2(x_1) = \mu_{M_1^2} = 1 - w_1(x_1)$. The stable reference model is in the form (6) with the following system and input matrices, and reference input vector, $\mathbf{A_r} = \begin{bmatrix} -1 & -1 \\ 0.25 & -10.5 \end{bmatrix}$, $\mathbf{B_r} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{r}(t) = 2\sin(5t)$.

With the chosen $\mathbf{C}$, based on (18), $\mathbf{\Gamma} = \text{diag}\{1, 1\}$ is obtained. Theorem 1 is employed to determine the system stability and feedback gains. To apply the third-party Matlab toolbox SOSTOOLS [37] for finding the solution numerically, we choose $\mathbf{X}(\tilde{\mathbf{x}})$, $\mathbf{M}_j(\mathbf{h})$ and $\mathbf{N}_j(\mathbf{h})$ for all j as constant matrices, $\varepsilon_1 = 0.001$ and $\varepsilon_2(\mathbf{w}) = 0.001(\bar{w}_1^2 + \bar{w}_2^2)^2$ (please note that $\bar{w}_1$ and $\bar{w}_2$ are taken as symbolic variables by the SOSTOOLS).

Choosing $\sigma_1 = 10$, $\sigma_2 = 10$, the degrees of the feedback gains and $\mathbf{X}(\tilde{\mathbf{x}})$ as both 0, the feedback gains are found as $\mathbf{F}_1 = -21.2611$, $\mathbf{F}_2 = -9.6202$, $\mathbf{G}_1 = 0.2119 \times 10^{-3}$ and $\mathbf{G}_2 = -0.1219 \times 10^{-3}$. The proposed fuzzy controller (11) is employed to control the nonlinear plant subject to the initial condition $\mathbf{x}(0) = [2\ 0]^T$ and $\mathbf{x_r}(0) = [0\ 0.05]^T$. The system state responses are shown in Fig. 1(a) and (b). Choosing $\sigma_1 = 7.5$ and $\sigma_2 = 0.1$ to demonstrate how they influence the H_∞ performance, the feedback gains are found as $\mathbf{F}_1 = -616.2673$, $\mathbf{F}_2 = -130.3563$, $\mathbf{G}_1 = 0.9833 \times 10^{-4}$ and $\mathbf{G}_2 = -0.5191 \times 10^{-4}$. The system responses subject to $\mathbf{x}(0) = [2\ 0]^T$ and $\mathbf{x_r}(0) = [0\ 0.05]^T$ are shown in Fig. 1(c) and (d). It can be seen that both fuzzy controllers with different values of σ_1 and σ_2 are able to handle the tracking control problem. However, the smaller the values of σ_1 and σ_2, the better the H_∞ tracking performance governed by (32) can be achieved.

For comparison purposes, we consider the T-S fuzzy model representing the nonlinear plant. It can be seen from the simulation results that x_1 is within -2 and 2. It is reasonable to consider the operating range of $x_1 \in [-2, 2]$ for the construction of T-S fuzzy model. Based on the sector nonlinearity concept, the T-S fuzzy model has eight rules with the system and input matrices as $\mathbf{A}_1 = \mathbf{A}_3 = \begin{bmatrix} 0.59 - 0.12a_{1_{\min}} & -7.29 - 1.82a_{1_{\min}} \\ 0.01 & -2.85 \end{bmatrix}$, $\mathbf{A}_2 = \mathbf{A}_4 = \begin{bmatrix} 0.02 + 2.25a_{1_{\min}} & -4.64 + 0.72a_{1_{\min}} \\ 0.35 & -8.56 \end{bmatrix}$, $\mathbf{A}_5 =$



(a) (b)

Fig. 2. (a) System responses of $x_1(t)$ (Solid line) and $x_{r_1}(t)$ (Dotted line). (b) System responses of $x_2(t)$ (Solid line) and $x_{r_2}(t)$ (Dotted line).

$\mathbf{A}_7 = \begin{bmatrix} 0.59 - 0.12a_{1_{\max}} & -7.29 - 1.82a_{1_{\max}} \\ 0.01 & -2.85 \end{bmatrix}$, $\mathbf{A}_8 = \begin{bmatrix} 0.02 + 2.25a_{1_{\max}} & -4.64 + 0.72a_{1_{\max}} \\ 0.35 & -8.56 \end{bmatrix}$; $\mathbf{B}_1 = \begin{bmatrix} 1+1.35a_{1_{\min}}+2.33a_{2_{\min}} \\ 0 \end{bmatrix}$, $\mathbf{B}_2 = \begin{bmatrix} 8-0.62a_{1_{\min}}+0.56a_{2_{\min}} \\ 0 \end{bmatrix}$, $\mathbf{B}_3 = \begin{bmatrix} 1+1.35a_{1_{\min}}+2.33a_{2_{\max}} \\ 0 \end{bmatrix}$, $\mathbf{B}_4 = \begin{bmatrix} 8-0.62a_{1_{\min}}+0.56a_{2_{\max}} \\ 0 \end{bmatrix}$, $\mathbf{B}_5 = \begin{bmatrix} 1+1.35a_{1_{\max}}+2.33a_{2_{\min}} \\ 0 \end{bmatrix}$, $\mathbf{B}_6 = \begin{bmatrix} 8-0.62a_{1_{\max}}+0.56a_{2_{\min}} \\ 0 \end{bmatrix}$, $\mathbf{B}_7 = \begin{bmatrix} 1+1.35a_{1_{\max}}+2.33a_{2_{\max}} \\ 0 \end{bmatrix}$, $\mathbf{B}_8 = \begin{bmatrix} 8-0.62a_{1_{\max}}+0.56a_{2_{\max}} \\ 0 \end{bmatrix}$, where $a_{1_{\max}} = -a_{1_{\min}} = 2$ and $a_{2_{\max}} = -a_{2_{\min}} = 4$. Based on this T-S fuzzy model, with the same settings above, no feasible control design can be found.

Remark 10: It can be seen from this example that the PFMB control approach demonstrates an enhanced feedback compensation capability with less number of rules. Furthermore, unlike the T-S fuzzy model, in this example, the polynomial fuzzy model is a global model without necessarily considering the operating region of x_1.

B. Simulation Example 2

An inverted pendulum on a cart is considered [17]. The dynamics of the system is described as follows:

$$\ddot{\theta}(t) = \frac{g\sin(\theta(t)) - am_pL\dot{\theta}(t)^2\sin(\theta(t))\cos(\theta(t))}{4L/3 - am_pL\cos^2(\theta(t))} - \frac{a\cos(\theta(t))\,u(t)}{4L/3 - am_pL\cos^2(\theta(t))} \quad (34)$$

where $\theta(t)$ is the angular displacement of the pendulum, $g = 9.8$ m/s^2 is the acceleration due to gravity, $m_p = 2$ kg is the mass of the pendulum, $M_c = 8$ kg is the mass of the cart, $a = 1/(m + M)$, $2L = 1$ m is the length of the pendulum, and $u(t)$ is the force applied to the cart.

Denoting $x_1(t)$ and $x_2(t)$ as $\theta(t)$ and $\dot{\theta}(t)$, respectively, the nonlinear plant can be represented by the following state-space equations:

$$\dot{x}_1(t) = x_2(t) \quad (35)$$

$$\dot{x}_2(t) = \frac{g\sin(x_1(t)) - am_pLx_2(t)^2\sin(x_1(t))\cos(x_1(t))}{4L/3 - am_pL\cos^2(x_1(t))} - \frac{a\cos(x_1(t))\,u(t)}{4L/3 - am_pL\cos^2(x_1(t))}. \quad (36)$$

In order to construct the polynomial fuzzy model for the inverted pendulum, we consider that the inverted pendulum is working in the operating domain of $x_1(t) \in [-(5\pi/12), \ (5\pi/12)]$ leading to $f_1(x_1(t)) = (\cos(x_1(t))/(4L/3 - am_pL\cos^2(x_1(t))) \in [f_{1_{min}}, \ f_{1_{max}}] = [0.3922, \ 1.7647]$. Approximating $\sin(x_1(t))$ and $\tan(x_1(t))$ by polynomials $\sin(x_1(t)) \approx s_3x_1(t)^3 + s_1x_1(t)$ and $\tan(x_1(t)) \approx t_3x_1(t)^3 + t_1x_1(t)$, respectively, where $s_3 = -0.1460$, $s_1 = 0.9897$, $t_3 = 1.0545$ and $t_1 = 0.6469$, the inverted pendulum (35) is described by a two-rule polynomial fuzzy model with the following system, input and output matrices, $\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x}$, $\hat{\mathbf{x}}_\mathbf{r}(\mathbf{x}_\mathbf{r}) = \mathbf{x}_\mathbf{r}$, $\mathbf{A}_1(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ a_1 & 0 \end{bmatrix}$, $\mathbf{A}_2(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ a_2 & 0 \end{bmatrix}$, $\mathbf{B}_1(\mathbf{x}) = \begin{bmatrix} 0 \\ -f_{1_{min}}a \end{bmatrix}$, $\mathbf{B}_2(\mathbf{x}) = \begin{bmatrix} 0 \\ -f_{1_{max}}a \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $a_1 = f_{1_{min}}(g(t_3x_1(t)^2 + t_1) - am_pLx_2(t)^2(s_3x_1(t)^2 + s_1))$, $a_2 = f_{1_{max}}(g(t_3x_1(t)^2 + t_1) - am_pLx_2(t)^2(s_3x_1(t)^2 + s_1))$. With the chosen $\mathbf{C}$, the polynomial fuzzy controller is a full-state feedback polynomial fuzzy controller.

The membership functions are defined as $\mu_{M_1^1}(x_1(t)) = w_1(x_1(t)) = (f_1(x_1(t)) - f_{1_{max}})/(f_{1_{min}} - f_{1_{max}})$ and $\mu_{M_2^1}(x_1(t)) = w_2(x_1(t)) = 1 - \mu_{M_1^1}(x_1(t))$, which can be obtained based on the sector nonlinearity concept [17] and [35]. The stable reference model is chosen as a linear system with $\mathbf{A}_\mathbf{r} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$, $\mathbf{B}_\mathbf{r} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{r}(t) = 5\sin(5)$. With the chosen $\mathbf{C}$, based on (18), it gives $\mathbf{\Gamma} = \mathrm{diag}\{1, 1\}$.

To lower the computational demand, Corollary 1 is employed to determine the system stability and feedback gains. Choosing $\mathbf{X}(\tilde{\mathbf{x}})$ as a constant matrix, $\mathbf{M}_j(\mathbf{h})$ and $\mathbf{N}_j(\mathbf{h})$ for all j as polynomial matrices with degree 4; $\varepsilon_1 = 0.001$ and $\varepsilon_2 = 0.001$; $\sigma_1 = 0.1$ and $\sigma_2 = 0.5$, the feedback gains are obtained as follows, $\mathbf{F}_1(\mathbf{x}(t)) = [1950.3226 + 1218.8707x_1(t)^2 + 726.3429x_2(t)^2 + 730.9874x_1(t)^2x_2(t)^2$ $1871.2828 + 1115.4328x_1(t)^2 + 727.2748x_2(t)^2 + 730.7731x_1(t)^2x_2(t)^2]$, $\mathbf{F}_2(\mathbf{x}(t)) = [774.1495 + 507.4451x_1(t)^2 + 361.6092x_2(t)^2 + 363.5183x_1(t)^2x_2(t)^2$ $705.2619 + 404.0699x_1(t)^2 + 362.5726x_2(t)^2 + 363.3356x_1(t)^2x_2(t)^2]$, $\mathbf{G}_1(\mathbf{x}(t)) = [134.3825 + 103.3143x_1(t)^2 - 0.9941x_2(t)^2 + 0.1514x_1(t)^2x_2(t)^2$ $70.9777 - 0.0275x_1(t)^2 + 0.0005x_2(t)^2 - 0.0001x_1(t)^2x_2(t)^2]$, $\mathbf{G}_2(\mathbf{x}(t)) = [85.4324 + 103.3445x_1(t)^2 - 0.9943x_2(t)^2 +$

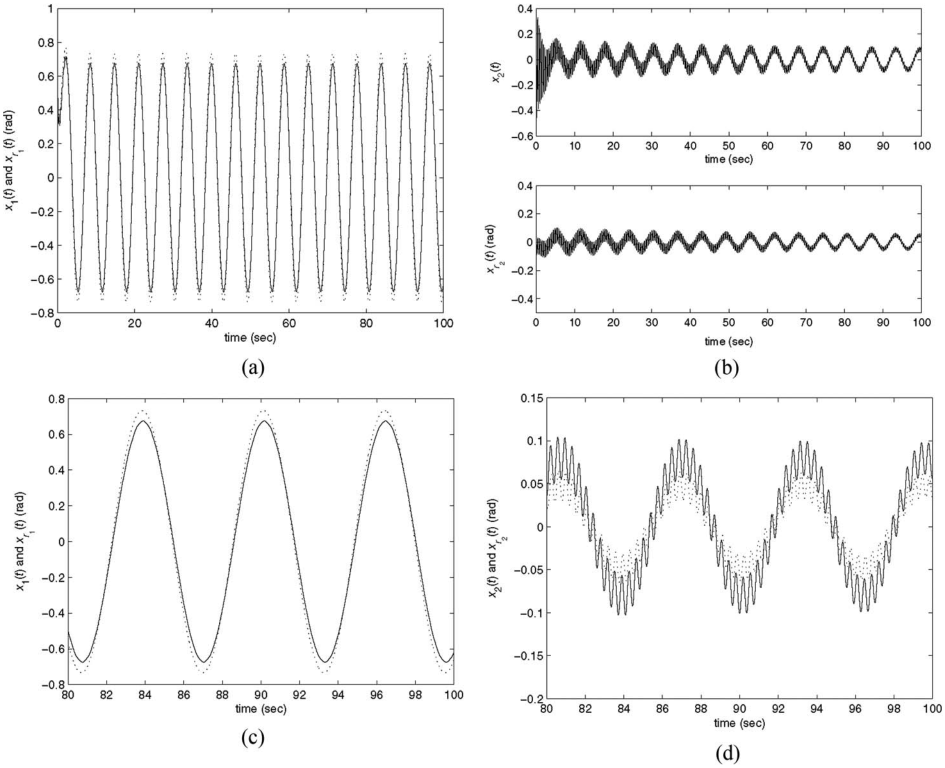


Fig. 3. (a) $x_1(t)$ (Solid line) and $x_{r_1}(t)$ (Dotted line) for $0 \leq t \leq 100$ s. (b) $x_1(t)$ (Solid line) and $x_{r_1}(t)$ (Dotted line) $80 \leq t \leq 100$ s. (c) $x_2(t)$ and $x_{r_2}(t)$ for $0 \leq t \leq 100$ s. (d) Solid line) and $x_{r_2}(t)$ (Dotted line) $80 \leq t \leq 100$ s.

$0.1514x_1(t)^2x_2(t)^2$ $22.0503+0.0041x_1(t)^2+0.0002x_2(t)^2-0.0001x_1(t)^2x_2(t)^2]$.

The polynomial fuzzy controller is employed to control the nonlinear plant subject to the initial condition $\mathbf{x}(0) = [5\pi/12\ 0]^T$ and $\mathbf{x_r}(0) = [5\pi/24\ 0.05]^T$. The system responses are shown in Fig. 2(a) and (b). It can be seen from the figures that the system states of the inverted pendulum are able to follow those of the stable reference model.

Remark 11: In this example, it demonstrates the way constructing the polynomial fuzzy model based on a given nonlinear mathematical model. Also, by choosing the matrix $\mathbf{C}$ as the identify matrix, the proposed output-feedback polynomial fuzzy controller becomes a full-state feedback one.

C. Simulation Example 3

A single-link flexible joint manipulator operating on a vertical plane [52] is considered. The system dynamics of the system is described by the following state-space equations:

$$\dot{x}_1(t) = x_3(t) \tag{37}$$

$$\dot{x}_2(t) = x_4(t) \tag{38}$$

$$\dot{x}_3(t) = \frac{K_s}{J_h}x_2(t) - \frac{K_m^2K_g^2}{R_mJ_h}x_3(t) + \frac{K_mK_g}{R_mJ_h}u(t) \tag{39}$$

$$\begin{aligned}\dot{x}_4(t) = &-\frac{K_s}{J_h}x_2(t) + \frac{K_m^2K_g^2}{R_mJ_h}x_3(t) - \frac{K_mK_g}{R_mJ_h}u(t)\\ &-\frac{K_s}{J_l}x_2(t) + \frac{m_lgh}{J_l}\sin(x_1(t)+x_2(t))\end{aligned} \tag{40}$$

where $x_1(t)$ denotes the angular position of the motor, $x_2(t)$ denotes the angular displacement of the flexible joint, $x_3(t)$ denotes the angular velocity of the motor and $x_4(t)$ denotes the angular velocity of the flexible joint; $K_s = 1.61$ (N/m) is the spring stiffness, $J_h = 0.0021$ (Kgm^2) is the inertia of hub, $m_l = 0.403$ (Kg) is the link mass, $g = -9.81$ (N/m) is the gravity constant, $h = 0.06$ (m) is the height of center of gravity, $K_m = 0.00767$ (N/rad/s) is the motor constant, $K_g = 100$ is the gear ratio, $J_l = 0.0059$ (Kgm^2) is the load inertia, and $R_m = 0.1$ (Ω) is the motor resistance.

Considering $x_1(t) + x_2(t) \in [-(\pi/2), (\pi/2)]$ leading to $f_1(x_1(t)) = ((\sin(x_1(t)+x_2(t)))/(x_1(t)+x_2(t))) \in [f_{1_{min}},\ f_{1_{max}}] = [0,\ 1]$, the single-link flexible joint manipulator can be represented by a two-rule T-S fuzzy model with the following system, input and output matrices:

$$\hat{\mathbf{x}}(\mathbf{x}) = \mathbf{x}, \hat{\mathbf{x}}_\mathbf{r}(\mathbf{x_r}) = \mathbf{x_r}$$

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{K_s}{J_h} & -\frac{K_m^2K_g^2}{R_mJ_h} & 0 \\ f_{1_{min}}\frac{m_lgh}{J_l} & a_1 & \frac{K_m^2K_g^2}{R_mJ_h} & 0 \end{bmatrix}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{K_s}{J_h} & -\frac{K_m^2K_g^2}{R_mJ_h} & 0 \\ f_{1_{max}}\frac{m_lgh}{J_l} & a_2 & \frac{K_m^2K_g^2}{R_mJ_h} & 0 \end{bmatrix}$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{K_mK_g}{R_mJ_h} \\ -\frac{K_mK_g}{R_mJ_h} \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{K_mK_g}{R_mJ_h} \\ -\frac{K_mK_g}{R_mJ_h} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

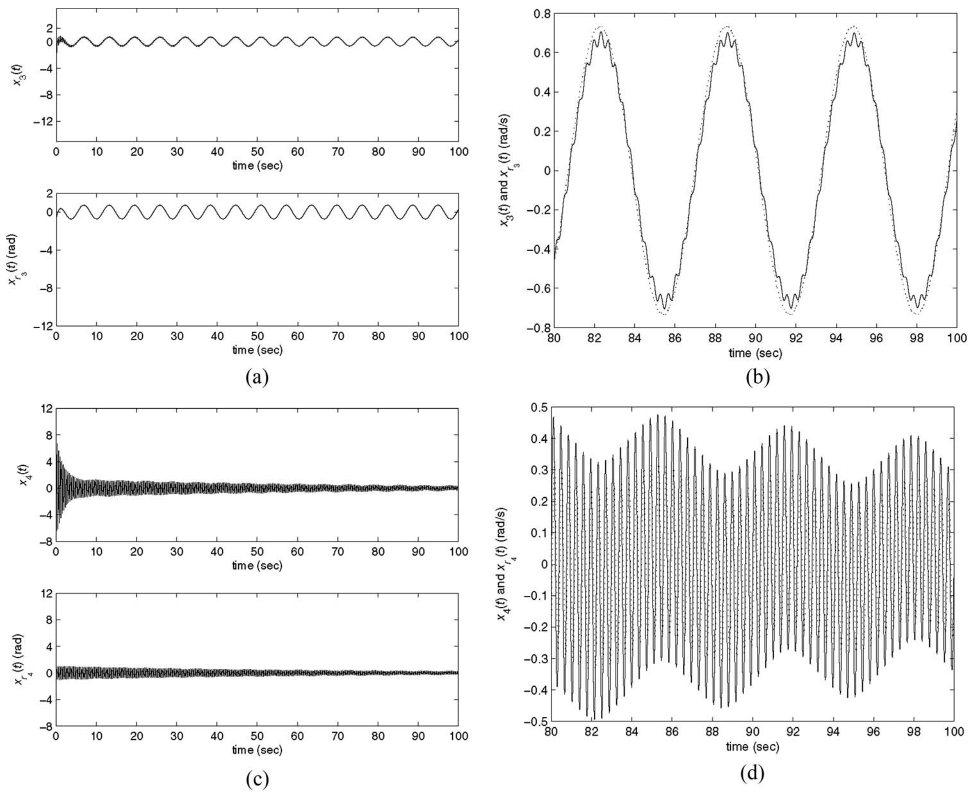


Fig. 4. (a) $x_3(t)$ and $x_{r_3}(t)$ for $0 \leq t \leq 100$ s. (b) $x_3(t)$ (Solid line) and $x_{r_3}(t)$ (Dotted line) $80 \leq t \leq 100$ s (c) $x_4(t)$ and $x_{r_4}(t)$ for $0 \leq t \leq 100$ s. (d)System responses of $x_4(t)$ (Solid line) and $x_{r_4}(t)$.

with

$$a_1 = -\left(\frac{K_s}{J_h} + \frac{K_s}{J_l}\right) + f_{1_{min}} \frac{m_l g h}{J_l}$$

$$a_2 = -\left(\frac{K_s}{J_h} + \frac{K_s}{J_l}\right) + f_{1_{max}} \frac{m_l g h}{J_l}.$$

The membership functions are defined as $\mu_{M_1^1}(x_1(t)) = w_1(x_1(t)) = ((f_1(x_1(t)) - f_{1_{max}})/f_{1_{min}} - f_{1_{max}}))$ and $\mu_{M_2^1}(x_1(t)) = w_2(x_1(t)) = 1 - \mu_{M_1^1}(x_1(t))$, which can be obtained based on the sector nonlinearity concept [17] and [35]. The stable reference model is chosen as a linear system with

$$\mathbf{A_r} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4134.2746 & 145.7871 & -2761.0653 & 40.4646 \\ 4114.1724 & -438.7707 & 2761.0653 & -40.4647 \end{bmatrix}$$

(with eigenvalues of $-0.0150 + 17i$, $-0.0150 - 17i$, -1.5, and -2800)

$$\mathbf{B_r} = \begin{bmatrix} 0 \\ 0 \\ \frac{K_m K_g}{R_m J_h} \\ -\frac{K_m K_g}{R_m J_h} \end{bmatrix}$$

$$\mathbf{r}(t) = \sin(t).$$

With the chosen $\mathbf{C}$, based on (18), $\mathbf{\Gamma} = \text{diag}\{1, 1, 1, 1\}$ is obtained. Choosing $\mathbf{X}(\tilde{\mathbf{x}})$, $\mathbf{M}_j(\mathbf{h})$ and $\mathbf{N}_j(\mathbf{h})$ for all j as constant matrices; $\varepsilon_1 = 0.001$ and $\varepsilon_2 = 0.001$; $\sigma_1 = 0.1$ and $\sigma_2 = 10$, with Corollary 1, the feedback gains are found as $\mathbf{F}_1 = [-15.3296\ 2.1752]$, $\mathbf{F}_2 = [-15.3368\ 2.1694]$, $\mathbf{G}_1 = [-1.1285\ -0.1672]$ and $\mathbf{G}_2 = [-1.1359\ -0.1730]$.

The fuzzy controller is employed to control the single-link flexible joint manipulator subject to the initial condition $\mathbf{x}(0) = [1\ 0\ 0\ 0]^T$ and $\mathbf{x_r}(0) = [0.5\ 0\ 0\ 0]^T$. The system state responses are shown in Figs. 3 and 4. It can be seen that the fuzzy controller is able to drive the system states to following those of the stable reference model.

Remark 12: In this example, all system and input matrices, and feedback gains are constant. The PFMB control system becomes a traditional FMB control system consisting of the T-S fuzzy model and fuzzy controller. It can be seen that the SOS-based stability conditions can also be applied to the traditional T-S fuzzy model.

IV. Conclusion

The tracking control problem of PFMB control systems has been investigated. An output-feedback polynomial fuzzy controller has been proposed to drive the system states of the nonlinear plant to follow those of the stable reference model subject to an H_∞ performance. The stability of the PFMB control system has been investigated based on the Lyapunov stability theory. SOS-based stability conditions have been derived to guarantee the system stability and facilitate the stable design of the output-feedback polynomial fuzzy controller. By using the third-party Matlab toolbox, SOSTOOLS, a feasible solution to the SOS-based stability conditions can be found numerically.

Simulation examples have been given to demonstrate the merits of the proposed approach. In the future, the proposed output-feedback PFMB control approach can be extended to different classes of nonlinear systems, e.g., sampled-data and time-delay systems.

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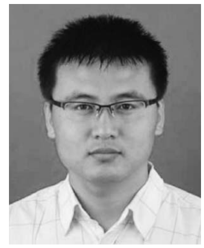

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