ε_1 denotes those strains over and above the mechanically induced elastic strains.

$$\varepsilon_1 = \varepsilon_p + \varepsilon_c + \varepsilon_T + \cdots$$
$$\varepsilon_p = \text{instantaneous plastic strain}$$
$$\varepsilon_c = \text{creep strain}$$
$$\varepsilon_T = \text{thermal strain}$$

The plastic and creep strain components may be derived from suitable constitutive laws of the types discussed in Chapter 2, while thermal strains are simply related to the temperature distribution in the structure.

The first example known to us of the use of this method is due to Goodey [4], who examined the case of stress redistribution resulting from creep in a beam, subjected to a constant bending moment. This case is dealt with first, but it is extended here to the more interesting case of a beam loaded by an axial force, together with a bending moment. The results of such an analysis have been used to investigate the effects of nonaxial loading during the tensile creep test [17] (Appendix 1).

3.2.1 The one-dimensional case of beam bending

The analysis is developed for a section having symmetry about one principal axis of bending (Fig. 3.3).

The total strain

$$\begin{aligned} \varepsilon &= \varepsilon_e + \varepsilon_1 \\ & \frac{\sigma}{E} + \varepsilon_1 \text{ (from the use of Hooke's law)} \end{aligned} \tag{3.1}$$

where $\sigma = \sigma(z)$ is the normal stress at height z from the centroidal plane (Fig. 3.2).

From the fact that originally plane sections remain plane after bending

$$\varepsilon = \kappa z + \hat{\varepsilon} \tag{3.2}$$

where κ is the curvature change and $\hat{\varepsilon}$ the strain at the centroidal plane.

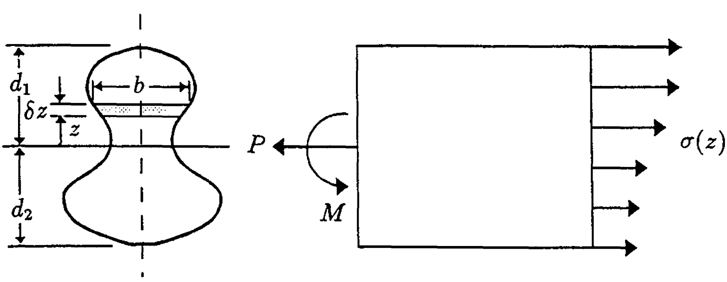

Fig. 3.3 Geometry of a beam section subjected to end load and bending moment.

Eliminating ε from (3.1) and (3.2) and solving for σ gives

$$\sigma = E\kappa z + E(\hat{\varepsilon} - \varepsilon_1) \tag{3.3}$$

Equilibrium of the section requires

$$P = \int_{-d_2}^{d_1} \sigma b \, \mathrm{d}z; \quad M = \int_{-d_2}^{d_1} \sigma b z \, \mathrm{d}z$$

and substituting for σ from (3.3) into the equilibrium equations gives

$$\left.\begin{aligned} P &= EA\hat{\varepsilon} - \int_{-d_2}^{d_1} E\varepsilon_1 b \, \mathrm{d}z \\ M &= EI\kappa - \int_{-d_2}^{d_1} E\varepsilon_1 bz \, \mathrm{d}z \end{aligned}\right\} \tag{3.4}$$

A and I are the section area and second moment of area respectively.

Equations (3.4) are seen to differ from the usual elastic ones, by the inclusion of the extra terms involving the inelastic strain terms ε_1 on their right-hand sides. This is a feature of solving creep problems, which will be consistently apparent in this chapter; the governing equations for the creep case are precisely the same in *form* as for the elastic problem.

As well as forming equations (3.4), it is necessary to derive their rate versions in order to gain complete solution. This is achieved by straight-forward differentiation of equation (3.4) with respect to time – a procedure which is admissible, provided that the deflections are small. The rate versions of equations (3.2), (3.3) and (3.4) become

$$\left.\begin{aligned} \varepsilon' &= \kappa' z + \hat{\varepsilon} \\ \sigma' &= E\kappa' z + E(\hat{\varepsilon}' - \varepsilon_1') \end{aligned}\right\} \tag{3.5}$$

$$\left.\begin{aligned} P' &= EA\hat{\varepsilon}' - \int_{-d_2}^{d_1} E\varepsilon_1' b \, \mathrm{d}z \\ M' &= EI\kappa' - \int_{-d_2}^{d_1} E\varepsilon_1' bz \, \mathrm{d}z \end{aligned}\right\} \tag{3.6}$$

where $(..)' = \partial(..)/\partial t$ denotes differentiation with respect to time.

Enough equations are now available for complete solution, provided knowledge of the strain component ε_1 is available.

To simplify the discussion at this stage, the following conditions are chosen as appropriate to a creeping material following instantaneous elastic response

$$\varepsilon_1 = 0 \text{ at } t = 0$$

$$\frac{\mathrm{d}\varepsilon_c}{\mathrm{d}t} = \varepsilon_c' = \text{creep strain rate} = f_1(\sigma)\frac{\mathrm{d}f_2(t)}{\mathrm{d}t} \text{ for } t \geqslant 0$$

and, taking the particular case $f_1(\sigma) = \sigma^m$ as advocated in Chapter 2 as a particularly useful form, this becomes

$$\varepsilon_c' = \sigma^m f_2'(t)$$

Further, in accordance with our initial assumptions regarding the history of loading, $P' = M' = 0$.

We then have from (3.2), (3.3), (3.4), (3.5) and (3.6)

$$\left.\begin{aligned} \varepsilon &= \kappa z + \hat{\varepsilon} \\ \sigma &= E\kappa z + E\hat{\varepsilon} \\ \hat{\varepsilon} &= \frac{P}{EA} \\ \kappa &= \frac{M}{EI} \end{aligned}\right\} t = 0 \tag{3.7}$$

$$\left.\begin{aligned} \varepsilon' &= \kappa' z + \hat{\varepsilon}' \\ \sigma' &= E\kappa' z + E(\hat{\varepsilon}' - \varepsilon_c') \\ \hat{\varepsilon}' &= \frac{1}{A}\int_{-d_2}^{d_1} \varepsilon_c' b \, dz \\ \kappa' &= \frac{1}{I}\int_{-d_2}^{d_1} \varepsilon_c' bz \, dz \end{aligned}\right\} t \geqslant 0 \tag{3.8}$$

Equations (3.7) are immediately recognizable as the usual elastic results. All are conveniently nondimensionalized in the following way for efficient computation

$$\Sigma = \frac{\sigma}{\sigma_0}, \quad \lambda = \frac{\varepsilon}{\varepsilon_0}, \quad \sigma_0 = E\varepsilon_0, \quad \eta = \frac{b}{d_2}, \quad \xi = \frac{z}{d_2}$$

where σ_0 is a reference stress to be chosen from convenience, e.g.

$$\sigma_0 = \frac{Md_2}{I}$$

Then (3.7) and (3.8) become

$$\left.\begin{aligned} \lambda &= \Sigma \\ \Sigma &= \xi + \frac{P}{A\sigma_0} \end{aligned}\right\} \tau = 0 \tag{3.9}$$

$$\left.\begin{aligned} \dot{\lambda} &= \dot{\Sigma} + \Sigma^m \\ \dot{\Sigma} &= \alpha \int_{-1}^{d_2/d_1} \Sigma^m \eta \, d\xi + \beta\xi \int_{-1}^{d_2/d_1} \Sigma^m \eta \xi \, d\xi - \Sigma^m \end{aligned}\right\} \tau \geqslant 0 \tag{3.10}$$

where

$$\alpha = \frac{d_2^2}{A}, \quad \beta = \frac{d_2^4}{I}$$

$$(\,\dot{}\,) = \frac{\partial(\,)}{\partial\tau}$$

and

$$\tau = E\sigma_0^{m-1}\int_{p=0}^{p=t} f_2'(p)\mathrm{d}p = E\sigma_0^{m-1}f_2(t). \tag{3.11}$$

is a nondimensional time parameter which will appear in all other problems to be discussed.

Enough equations have now been assembled to attempt solution to the problem. For a given geometry of beam section ($b(\xi), d_1, d_2$) and values of the loading (M, P), equations (3.9) enable the elastic stresses and strains at any section (ξ) to be calculated at time (τ) zero. Knowing the stresses at zero time the integrals in (3.10) can be evaluated so that the stress and strain rates at time zero are also determinate. Assuming that these rates hold over a small time interval $\Delta\tau$, then the stresses and strains at the end of the time intervals can be determined from

$$\Sigma_{\tau=\Delta\tau} = \Sigma_{\tau=0} + \dot{\Sigma}_{\tau=0} \times \Delta\tau + O(\ddot{\Sigma}_{\tau=0}, \Delta\tau^2, \text{etc.})$$

and likewise

$$\lambda_{\tau=\Delta\tau} = \lambda_{\tau=0} + \dot{\lambda}_{\tau=0} \times \Delta\tau + O(\ddot{\lambda}_{\tau=0}, \Delta\tau^2, \text{etc.})$$

The errors in these equations could be reduced by including terms in higher derivatives of the stresses and strains, but experience has shown this to be unnecessary, provided that a sensible choice of the time interval $\Delta\tau$ is made; guidance on this choice is given later. Once the decision has been made on the size of $\Delta\tau$, the now-known stresses at the new time $\Delta\tau$ permit calculation of new stress and strain rates at that time. Forward integration (in time) gives the new values of stresses and strains. This process is continued until the stresses change by no more than a specified limit (nominally zero), at which time the strain rates will be nominally constant. This state we call the *stationary state* at which time stress redistribution is complete. Computed values of the stress distribution at various times after elastic loading and up to the stationary state are illustrated in Fig. 3.4, for a rectangular section beam and the stress exponent $m = 3$. Appendix 4 contains a complete step-by-step description of this analysis.

There are few problems for which the stresses at the stationary state can be determined analytically, but the beam is one of them – the results for which can be derived from the preceding equations. This is given now by way of illustration, but as a digression from the main theme.

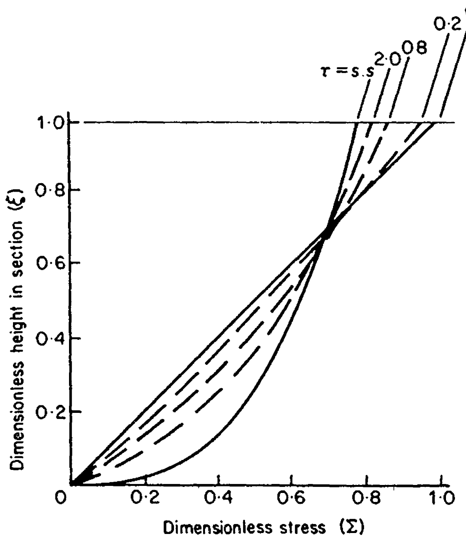

Fig. 3.4 Transient stress distribution in a rectangular section beam during pure bending (stress exponent $m = 3$).

For a beam in pure bending, equation (3.10) gives for the stress rate

$$\dot{\Sigma} = \beta\xi \int_{-1}^{d_2/d_1} \Sigma^m \eta \xi \, d\xi - \Sigma^m$$

and this rate is zero at the stationary state (ss). Thus

$$\Sigma_{ss}^m = \beta\xi \int_{-1}^{d_2/d_1} \Sigma_{ss}^m \eta \xi \, d\xi$$

In the case of a rectangular section beam $d_2 = d_1$, $\beta = 3d/2b$ and then

$$\Sigma_{ss}^m = \xi \int_{-1}^{1} \Sigma_{ss}^m \xi \, d\xi$$

But since $\Sigma_{ss} = \Sigma_{ss}(\xi)$ the integral in this equation is a constant $= C_1$ (say) Therefore

$$\Sigma_{ss} = (C_1 \xi)^{1/m} = K\xi^{1/m}$$

But the equilibrium equation requires that

$$\int_{-1}^{1} \Sigma_{ss} \xi \, d\xi = \frac{2}{3}$$

Therefore

$$\int_{-1}^{1} K\xi^{1/m} \xi \, d\xi = \frac{2}{3}$$

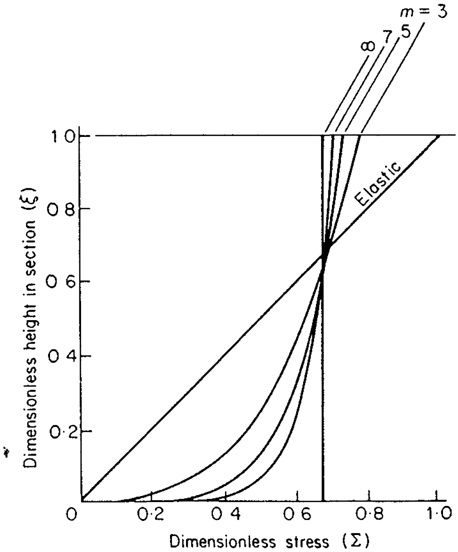

Fig. 3.5 Stationary state stress variations with stress exponent for a rectangular section beam during pure bending.

Therefore

$$K = \frac{2m+1}{3m}$$

The stationary state stress distribution is then given by

$$\Sigma_{ss} = \left(\frac{2m+1}{3m}\right)\xi^{1/m} \tag{3.12}$$

This distribution of stresses is shown in Fig. 3.5 for various values of m, the stress exponent. Converted to dimensional form, equation (3.12) becomes

$$\sigma_{ss} = \left(\frac{2m+1}{3m}\right)\frac{M}{bd^2}\left(\frac{z}{d}\right)^{1/m}$$

Summary of computational procedure

The method described is clearly well suited to digital computation and most of the problems which have been dealt with have been organized this way. At this stage it is worth summarizing the steps in the calculation procedure.

1. Solve the initial problem (elastic), equations (3.9).
2. Use the stresses from step 1 to solve the rate problem, equations (3.10).
3. Evaluate stress rates, etc., using step 2.

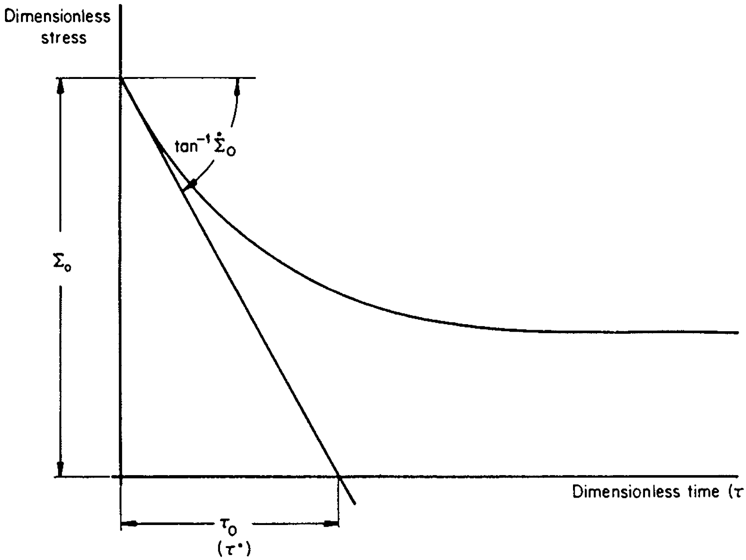

Fig. 3.6 Characteristics of stress redistribution.

4. Over a chosen time interval evaluate new stresses and any other quantities required from

$$\Sigma_{\tau+\Delta\tau} = \Sigma_\tau + \dot{\Sigma}_\tau \times \Delta\tau, \text{ and the like.}$$

5. Repeat steps 2–4 to any time required.

The question of choice of $\Delta\tau$ is helped by consideration of Fig. 3.6 which shows schematically the redistribution of stress at a given position in the beam section, from its initial elastic value Σ_0 at $\tau = 0$ to the stationary state. At $\tau = 0, \Sigma_0$ and $\dot{\Sigma}_0$ are known precisely so that the intercept of the line $\Sigma = \Sigma_0 + \dot{\Sigma}_{\tau_0}$ with the abscissa defines the point $\tau = \tau_0$ precisely. Clearly, it would be inadvisable to make $\Delta\tau \geqslant \Sigma_0/\dot{\Sigma}_0(=\tau_0)$, because this would lead to absurdly hasty redistribution. A better choice is $\Delta\tau = \tau_0/f$ where $f \gg 1$. Here f is a number which can be experimented with, in that it could be progressively changed, if one were uncertain of a correct choice, until solutions no longer changed with increasing values of f. For most problems $f = 5$ has been found to be adequate. The great advantage of choosing $\Delta\tau$ in this manner, is that it can be used throughout the calculation by taking local values of Σ and $\dot{\Sigma}$ with the effect of lengthening the time steps as time progresses. Thus, at $\tau = 0$ when stresses are changing rapidly, small time intervals are needed, but as time progresses towards the stationary state, such small steps are not necessary. The simple rule used at all stages of the computation is thus

$$\Delta\tau = \frac{1}{f}\left|\frac{\Sigma(\xi)}{\dot{\Sigma}(\xi)}\right|, \quad f \gg 1$$

In addition to choosing the time interval, it is also necessary to decide when to stop the computations. The end point in the time integration could be chosen at will, of course, but generally speaking, the point of most interest is when the stresses are stationary, i.e. $\dot{\Sigma} = 0$. The time at which stationarity is achieved is difficult to define and indeed complete stationarity may never be achieved. For the purpose of defining a state of stationarity numerically, tolerances on $\dot{\Sigma}$ must be specified. The limit $|\dot{\Sigma}| \leqslant 10^{-2}$ seems to be sufficient for this purpose, although it would be equally easy to place a limit on the stress level during relaxation as a definition, or on the constancy of strain rate to within a given tolerance. The definition given will depend on the context of the problem under investigation.

These guidelines are just that – guides. Anyone phrasing his own computations will find them useful and may generate his own criteria for choosing $\Delta\tau$ and a suitable state of stationarity. The object though is always the same: to avoid numerical instability and to minimize computing time. Most commercial programs dealing with creep contain effective time integration algorithms of one sort or another.

In order to illustrate further the numerical solving technique, a common two-dimensional problem involving a stress raiser will be outlined in the next section. An additional feature revealed in solving this problem is a demonstration that in certain circumstances, the stationary state solution gives the same results as those for time-independent plasticity.

3.2.2 The two-dimensional case of plate stretching [24]

All the usual assumptions of plane-stress are made: the plate is thin enough to make variations in stresses through the thickness negligible; only two principal components of stress, σ_r, σ_θ, are present in the axially symmetric problem (Fig. 3.7); strains are small (less than about 2%).

The loading is uniformly applied at the outer edge of the plate and, in the present example, the plate contains a hole at its centre which is free from any loading.

In common with the elastic problem the radial and circumferential stresses must satisfy the equilibrium equation

$$\frac{\partial}{\partial r}(r\sigma_r) - \sigma_\theta = 0 \tag{3.13}$$

and, as with the one-dimensional case, the total strains are considered as being composed of two parts

$$\begin{aligned} \varepsilon_r &= \varepsilon_r \text{ (elastic)} + \varepsilon_{r,c} \\ &= \frac{\sigma_r - \nu\sigma_\theta}{E} + \varepsilon_{r,c} \text{ from Hooke's law} \end{aligned} \tag{3.14}$$