Chapter 9
Material Response: Measures of Stress and Strain

Abstract Amongst the many pillars upon which the FEM solutions stand is the pillar of *material response*. This defines the physical behaviour of the material type under investigation in the FEM problem. It is under this pillar that one distinguishes between rubbery, elastic, nonlinear, fracture and interface failure mechanisms. It is a key component of the FEM problem and must be correctly defined if one is to obtain reliable solutions. A common theme for this pillar of the FEM process is what is described as *predictive modelling*, which is the use of computational methods to determine the material behaviour of a given material. In this chapter, we have presented the principles of the material response module of an FEM scheme. The focus here is exploiting the principles of continuum mechanics in defining the response of a material body. Specifically, this chapter introduces the kinematics of finite deformation of a material body; measures of strains and stresses; and concludes with the practical formulations of stresses needed by engineers during the design process. Such practical stress formulations include: principal stresses, von Mises stresses, etc. This chapter lays down the theory needed to understand the material model implementations in the finite element solver.

Keywords Material response • Finite deformation • Deformation gradient tensor • Strain tensor • Strain tensor • Stress measures

9.1 Introduction

Consider a virtual domain, shown in Fig. 9.1, bounded by Ω_{before} which represents the boundaries of a given material system *before* deformation. Let us impose a distributed load, $\boldsymbol{F}$, on the left hand side of the structure such that we can identify three possible deformation profiles. Whilst *Materials 1* and *2* are dominated by a form of compressive deformation, *Material 3* shows a dominant shear deformation. The material system is bounded by a virtual domain *after* the deformation represented by Ω_{after}. This deformed domain is different for all three material types. The answer to why this is the case lies in an understanding of the material composition of all three material systems.

The differences in deformation profiles are quantified by constructing a material model to capture the individual composition of the three materials. It is the goal

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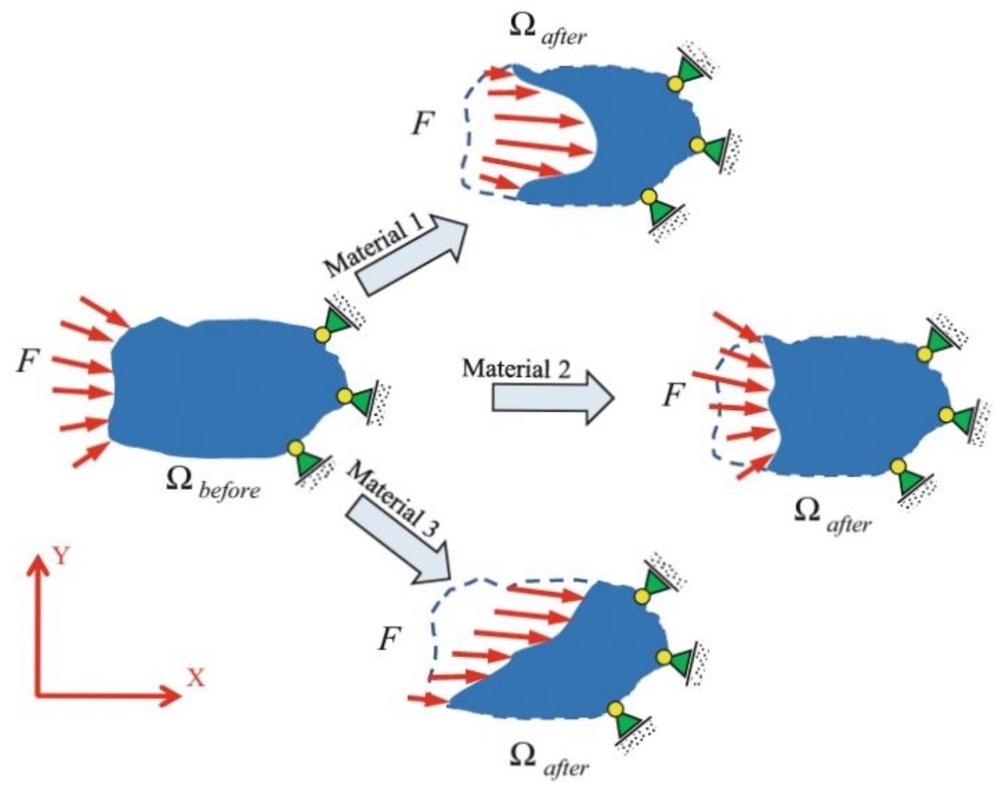

Fig. 9.1 Schematic representations of different material responses under the effect of a distributed load

of the FEM user to choose the correct material response to capture the expected material behaviour. The study of *material responses* here focuses on the numerical representation of the material behaviours that define how a structure can deform under the effect of load(s).

To describe the material response, we will seek to provide answers to the following questions.

- What are the reference frames that are required to describe the observed material response?
- What are the measures that describe the observed material response?
- What are the constitutive formulations that described the observed material response?
- How do FEM solvers implement the constitutive formulations into a material mode?
- What are the in-built material models in a typical FEM solver like `ABAQUS`?
- How can one extend the material models available in an FEM solver like `ABAQUS`?

In this chapter and the next, we will provide answers to these questions. The current chapter addresses the question of defining parameters needed to describe the deforming body. In the next chapter, the constitutive material models and their implementation within an FEM solver will be presented.

9.2 Chapter Objectives

At the end of this chapter, the reader should be able to:

(a) Define the kinematics of deformation needed in describing material response within the FEM process.
(b) Understand the different measures of representing the material response in an FEM scheme.
(c) Explore the different measures of strain needed to describe material response.
(d) Derive the different measures of stress that describe finite deforming systems.
(e) Introduce the different practical formulations of stresses needed by design in engineering to aid the design process for material systems subjected to multiple load histories.

9.3 Kinematics of Deformation

Continuum mechanics provides the fundamental principles for describing the material behaviours of different materials. Let us first isolate a *material body*, shown in Fig. 9.2, which represents either a form of solid, liquid or gas system. Consider the material body is bounded by a material domain, Ω, such that its microstructure consists of sets of elements, $\boldsymbol{X}$, called *particles* or *material points*.

On a reference frame of choice, *XYZ*, each of the material points can be related to a coordinate position on defined regions of the physical space. Each of these coordinate positions are measured with respect to the origin of the *XYZ* reference frame. The *kinematics of deformation* introduces the principles for describing the changes observed in material systems under the effect of loads.

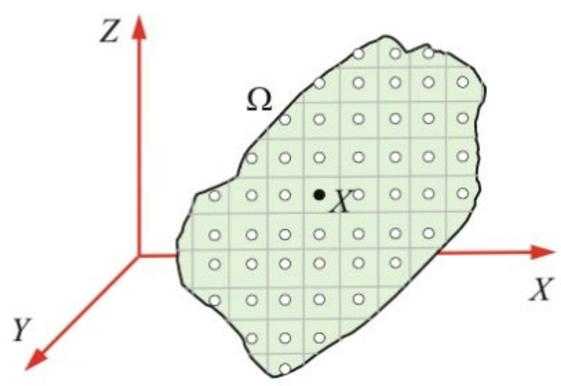

Fig. 9.2 A material body consisting of material points, $\boldsymbol{X}$, at defined coordinate positions

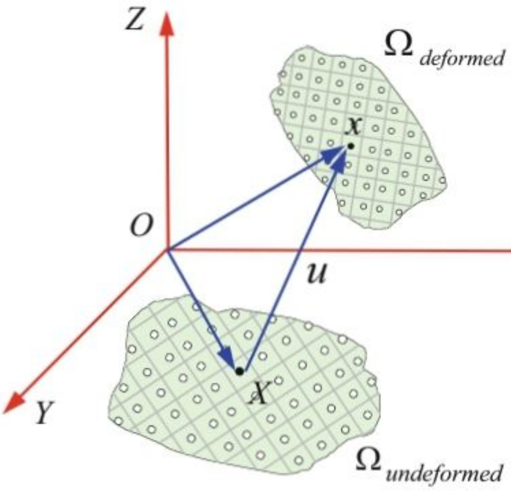

Fig. 9.3 A material body subjected to a displacement vector, $\boldsymbol{u}$, displacing it from its original (*material*) configuration, $\boldsymbol{X}$, to a new *spatial* configuration, $\boldsymbol{x}$

9.3.1 Material Configuration and Displacement

If we consider the material body of Fig. 9.3 to be a *rigid body*, the material points remain *locked* in their *original* coordinate positions during the deformation. However, for the purposes of defining material behaviour, we consider the material body to be *deformable*, i.e. it changes shape and size under the effect of loads. For a deformable body, the coordinate positions of the material points in Fig. 9.2 change in space and in time. At any given time, under the effect of external loads, the assembly of material points of the material body defines the *material configuration* of the body.

If the new configuration is defined by a new set of material points, $\boldsymbol{x}$, distinct from the initial reference configuration, $\boldsymbol{X}$, it is possible to mathematically relate the $\boldsymbol{x}$ material points to the original $\boldsymbol{X}$ material points according to the equations:

$$\boldsymbol{x} = \boldsymbol{\kappa}(\boldsymbol{X}) \qquad \text{and} \qquad \boldsymbol{X} = \boldsymbol{\kappa}^{-1}(\boldsymbol{x}) \tag{9.1}$$

where $\boldsymbol{\kappa}$ is the vector function that maps the $\boldsymbol{x}$ positions to the $\boldsymbol{X}$ positions. The above assumes that the mapping function is *invertible*. The above equation implies that the material points on a deformed material body can be related to their original positions and vice versa. It is this principle that drives the formulation of *stress* and *strain* measures that will be presented later.

Consider the material body experiences a change from its reference position, $\boldsymbol{X}$, to a new configuration, $\boldsymbol{x}$, as shown in Fig. 9.3. The change in configuration is caused by a *displacement vector*, $\boldsymbol{u}$, which consists of simultaneous *translation* and *rotation* of the material points relative to the origin of the fixed reference frame.

For a given displacement vector, $\boldsymbol{u}$, as shown in Fig. 9.3, the map of the material points at both original and deformed configurations becomes:

$$\boldsymbol{x} = \boldsymbol{\kappa}(\boldsymbol{X}) + \boldsymbol{u} \tag{9.2}$$

We will now extend the above discussion to account for the fact that the material points in the new configuration are a function not only of coordinate positions, but also time. Thus, Eq. 9.1 of the material body can be extended to incorporate time-dependence of the deformation, such that we write:

$$\boldsymbol{x} = \boldsymbol{\kappa}\left(\boldsymbol{X}, t\right) \qquad \text{and} \qquad \boldsymbol{X} = \boldsymbol{\kappa}^{-1}\left(\boldsymbol{x}, t\right) \tag{9.3}$$

9.3.2 *Material and Spatial Reference Frames*

In Continuum mechanics, two material configurations are essential in describing the kinematics of deformation of a material body. The first is the original configuration of the body before applying a displacement vector, called the *reference configuration*. It is a baseline set of material points of the material body from which subsequent measures of deformation are taken. From Eq. 9.3, this configuration coincides with time, $t = 0$. It is also the reference at which $\boldsymbol{u} = 0$, from Eq. 9.2, in which case: $\boldsymbol{x} = \boldsymbol{X}$.

Once time has elapsed, we define a new material configuration called the *current configuration*. From Fig. 9.3, the reference and current configurations are bounded by the $\Omega_{undeformed}$ and $\Omega_{deformed}$ boundary domains. It is customary in continuum mechanics to associate two coordinate systems to the two material configurations to quantitatively describe the kinematics of motion of the material body.

In the reference or original configuration, the *material reference frame* is used to describe the positions of the material points within the physical space. It is denoted by upper case letters. For example, for a Cartesian representation, the material reference frame is the $XYZ-$axes. The set of positions that describe a material body is represented in this frame by the vector, $\boldsymbol{X}$. Since the material coordinate is static and does not follow changing material configurations of the material body, it is said to be *time-independent*.

On the other hand, the *spatial* or *current reference frame* is a coordinate system that describes the current coordinate positions of a material body under the effect of external or internal loads. It is the cumulative location of the particles that make up the material body after each of the particles has been acted upon by a displacement vector, $\boldsymbol{u}$. It is represented by lower case letters. For the Cartesian coordinate representation, the spatial reference frame for a set of particles, $\boldsymbol{x}$, is the $xyz-$axes. The spatial coordinates are *time-dependent* as their values change with changing configurations of the material body.

We can illustrate these two reference frames by considering the fixed axis rotation of the bar shown in Fig. 9.4, rotating with angular velocity, ω. The bar is kept rotating by a perpendicular force vector, $\boldsymbol{F}$. The material reference frame is the XY-axes attached to the bar such that at all stages during the rotation, the axes rotate with the material. The material points of the body, with respect to the XY-axes, will remain the same at all three states of motion. The spatial reference frame is the xy-axes. It is similar to the global reference frame. With changing positions during motion of the bar, the material points coordinate positions will be changing with respect to this spatial (xy-axes) coordinate system.

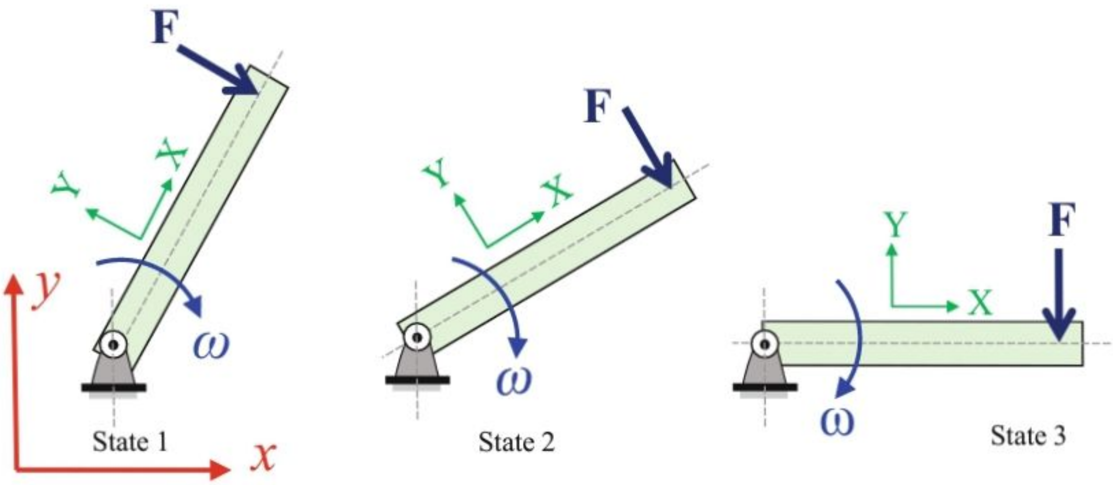

Fig. 9.4 An illustration of material (XY-) and spatial (xy-) reference frames for a fixed axis rotation of a slender bar during three states of the motion

When a physical property of a material body is expressed in terms of material coordinates, $\boldsymbol{X}$, and time, t, it is said to be given by *material*, or *Lagrangian, description*. In this description, the observer follows the material point as it moves along space and time. On the other hand, the *Eulerian*, or *spatial, description* describes the physical property in terms of the spatial coordinate system. This means the observer sits at a fixed point and assesses/observes the changes in configuration of the material body in space and time.

9.3.3 *Deformation Gradient*

Consider the material body of Fig. 9.2 being subjected to a displacement vector, $\boldsymbol{u}$. The material will experience deformation that comprises a change in shape and size. To quantify the deformation, let us isolate a line segment, $\overline{AB}$ of length, $\mathrm{d}\boldsymbol{X}$ in the material reference frame and $\overline{A'B'}$ of length $\mathrm{d}\boldsymbol{x}$ in the spatial reference frame, as shown in Fig. 9.5. The line $\overline{AB}$ is deformed, resulting in line $\overline{A'B'}$ such that all the material points that make up the material body will deform according to the equation: $\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}$.

We introduce the *deformation gradient tensor* (or, for short, *deformation gradient*), $\boldsymbol{F}$, to represent the mapping function that links the deformation of the material reference frame to that of the spatial reference frame. This implies that any infinitesimal deformation, $\mathrm{d}\boldsymbol{X}$ on the material reference frame is transformed to its equivalent spatial reference frame deformed state, $\mathrm{d}\boldsymbol{x}$, by the *deformation gradient*, and defined according to the equation:

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\mathrm{d}\boldsymbol{X} \qquad \Longrightarrow \qquad \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \frac{\partial\left(\boldsymbol{X}+\boldsymbol{u}\right)}{\partial \boldsymbol{X}} = \boldsymbol{I} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} \tag{9.4}$$

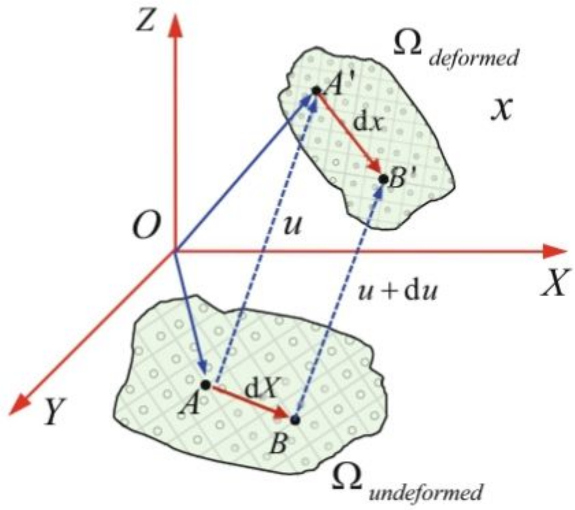

Fig. 9.5 An illustration of the deformation of a line segment, $\overline{AB}$ from material to spatial reference frames

where $\boldsymbol{I}$ is an identity matrix. The deformation gradient, $\boldsymbol{F}$, is the key parameter that defines the deformation of a material body. In its most general form (for a 3D system), $\boldsymbol{F}$ consists of nine components defined in terms of time, t. It is a second-order tensor and according to Eq. 9.4, the mapping function, $\boldsymbol{F}$, is a *linear* transformation function that maps $\mathrm{d}\boldsymbol{X}$ deformation to $\mathrm{d}\boldsymbol{x}$.

Using the left-hand side expression of Eq. 9.4, we can re-write it in component form for a Cartesian coordinate system representation in 3D of the material body thus:

$$\begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{bmatrix} = \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix} \begin{bmatrix} \mathrm{d}X \\ \mathrm{d}Y \\ \mathrm{d}Z \end{bmatrix} \tag{9.5}$$

The deformation gradient represents the behaviour of a material point with respect to another neighbourhood point, hence some authors describe it as a *two-point tensor* that links material points in two separate material configurations [5]. The deformation gradient can also be expressed in terms of the partial derivatives of the terms of the material points in the spatial reference frame thus:

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \quad \Longrightarrow \quad \boldsymbol{F} = \nabla \boldsymbol{x} \tag{9.6}$$

where $\nabla = \dfrac{\partial}{\partial \boldsymbol{X}}$ is the transpose of the vector differential operator. The component-wise expansion of Eq. 9.5 thus becomes:

$$\begin{bmatrix} \mathrm{d}x \\ \\ \mathrm{d}y \\ \\ \mathrm{d}z \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix} \begin{bmatrix} \mathrm{d}X \\ \\ \mathrm{d}Y \\ \\ \mathrm{d}Z \end{bmatrix} \tag{9.7}$$

Based on Eq. 9.3, let us assume mapping function, $\boldsymbol{\kappa}$, exists and also is invertible. The same philosophy will also hold for the mapping deformation gradient tensor, the material points have been displaced by a displacement vector, $\boldsymbol{u}$. As a result, we would expect there to exist an *inverse deformation gradient tensor*, $\boldsymbol{F}^{-1}$, which maps material points at the material reference frame, $\boldsymbol{X}$, to the spatial reference frame, $\boldsymbol{x}$. In other words, the inverse deformation gradient tensor transforms the spatial line segment, $\overline{A'B'}$ of Fig. 9.5 into the material line segment, $\overline{AB}$. Hence, the tensorial expression of the inverse deformation gradient is:

$$\boldsymbol{F}^{-1} = \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} \qquad \Longrightarrow \qquad \boldsymbol{F}^{-1} = \boldsymbol{\nabla}^{-1}\boldsymbol{x} \qquad \text{or} \qquad \mathrm{d}\boldsymbol{X} = \boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x} \tag{9.8}$$

9.3.4 Rotation and Stretch Tensors

Its important at this point to introduce two new terms associated with kinematics of deformation of a material body: *rotation* and *stretch*. Rotation is a measure of rigid body deformation of the body such that only angular changes occur during the deformation of the body without any change in size or shape. Stretch on the other hand refers to a change in size of the body independent of any changes in orientation. Generally, when a body deforms, both stretch and rotation measures are required to describe comprehensively the deformation of the body. These two terms arise due to a local motion of material points within a material body.

Consider a material body bounded by the $\Omega \subset \Re^n$ such that in its undeformed state, the boundary domain is bounded by $\Omega_{undeformed}$, while in the deformed state, the boundary domain is bounded by $\Omega_{deformed}$, as shown in Fig. 9.6. If the undeformed material body is subjected to a deformation gradient, $\boldsymbol{F}$, the resulting deformed configuration is given as a cumulative effect of *rotation* and *stretch*. The principle of *polar decomposition* is introduced to separate the contributions of the stretch and rotation tensors. The mapping deformation gradient, $\boldsymbol{F}(\boldsymbol{X}, t)$, can be decomposed into *pure stretch*, $\boldsymbol{U}$ or $\boldsymbol{V}$, and *pure rotation*, $\boldsymbol{R}$, components.

The first form of polar decomposition requires that the undeformed material body is first subjected to an orthogonal rotation tensor, $\boldsymbol{R}$, so that there is a resultant change of *local orientation*. $\boldsymbol{R}$ is a measure of this *local change in orientation*. After operating on the material body with the rotation tensor, the *intermediate deformed*

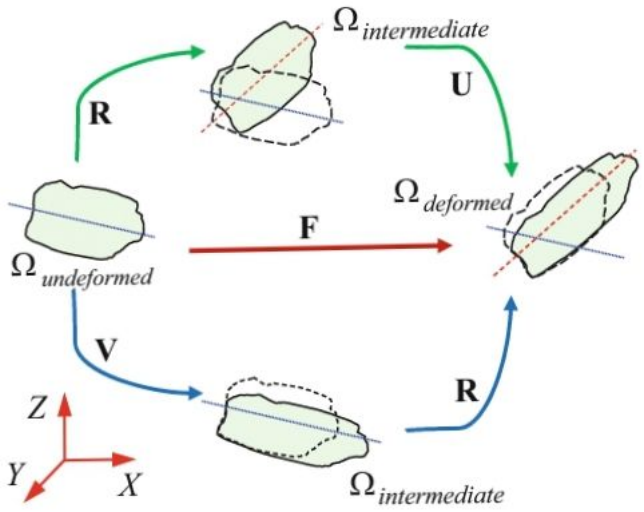

Fig. 9.6 An illustration of polar decomposition of the deformation gradient, $\boldsymbol{F}$, into rotation, $\boldsymbol{R}$, and stretch, $\boldsymbol{U}$ or $\boldsymbol{V}$, tensors

domain, $\Omega_{intermediate}$, is further operated by a *right* (or *material*) *stretch tensor*, $\boldsymbol{U}$, which introduces a local stretch (elongation or contraction) of $\Omega_{intermediate}$ along its main axial direction. This results in a *local change in size* without any change in orientation. Mathematically, the first form of polar decomposition is written as:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} \qquad \text{such that} \qquad \boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}, \quad \boldsymbol{U} = \boldsymbol{U}^T \quad \text{and} \quad \det(\boldsymbol{R}) = 1 \tag{9.9}$$

Similar to the first form of polar decomposition, we can also achieve the deformed configuration by first operating on $\Omega_{undeformed}$ with a *left* (or *spatial*) *stretch tensor*, $\boldsymbol{V}$, which causes a change in size of the material body resulting into the $\Omega_{intermediate}$ domain, as shown in Fig. 9.6. After that, the rotation tensor is operated on the intermediate domain, $\Omega_{intermediate}$, to achieve the deformed domain, $\Omega_{deformed}$. The mathematical expression on this second form of polar decomposition becomes:

$$\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R} \qquad \text{such that} \qquad \boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{V} = \boldsymbol{V}^T \tag{9.10}$$

9.3.5 *Velocity Gradient*

We established previously that the deformation of a material body is a function of not only displacement, but also time. The previous discussions of deformation gradient, stretch, rotation and strain measures considered a time-independent material body. It is important that for exhaustive an understanding of material responses within finite element modelling, the considerations of effect of time on the deformation need to be made. Plasticity formulations and viscoelasticity are ready examples to time-dependent material responses, hence need to be defined as rate quantities. The constitutive laws that describe material behaviour are often implemented within finite element modelling in their rate forms, hence the necessity of this section. We will now define time-dependent kinematics of deformation quantities.

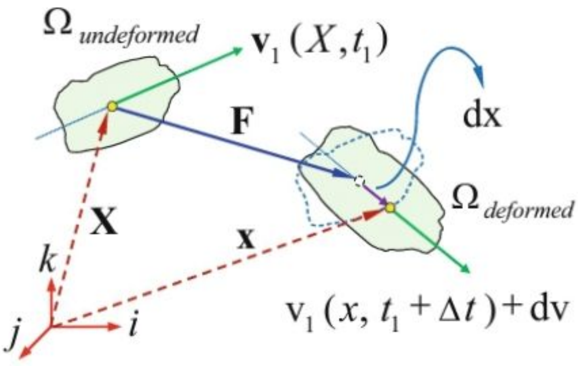

Fig. 9.7 An illustration of a material body with spatial varying velocity field

Consider a material body, $\Omega \subset \Re^n$, bounded by $\Omega_{undeformed}$ and $\Omega_{deformed}$ in the undeformed and deformed material configurations respectively and shown in Fig. 9.7. Let us assume that material point located at position $\boldsymbol{X}$ within $\Omega_{undeformed}$ has a velocity field, $\boldsymbol{v}_1$. If the material body is subjected to a deformation gradient, $\boldsymbol{F}$, for duration of Δt, the body rotates and elongates and the original material point now occupies a new position, $\boldsymbol{x}$, and the velocity field becomes $\boldsymbol{v}_1 + \mathrm{d}\boldsymbol{v}$.

Across the change of position, $\mathrm{d}\boldsymbol{x}$, of the material points in the deformed domain, the velocity field has changed by $\mathrm{d}\boldsymbol{v}$. The *velocity gradient*, $\boldsymbol{l}$, is the spatial rate of change of velocity field over the length, $\mathrm{d}\boldsymbol{x}$. The velocity gradient is derived by:

$$\mathrm{d}\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\mathrm{d}\boldsymbol{x} \qquad \text{where} \qquad \boldsymbol{l} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \quad = \begin{bmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{\partial v_x}{\partial y} & \dfrac{\partial v_x}{\partial z} \\[2ex] \dfrac{\partial v_y}{\partial x} & \dfrac{\partial v_y}{\partial y} & \dfrac{\partial v_y}{\partial z} \\[2ex] \dfrac{\partial v_z}{\partial x} & \dfrac{\partial v_z}{\partial y} & \dfrac{\partial v_z}{\partial z} \end{bmatrix} \tag{9.11}$$

Let us also relate the velocity gradient, $\boldsymbol{l}$, to the deformation gradient, $\boldsymbol{F}$, that caused the deformation of the material domain. Here, we consider the *rate of change of deformation gradient*, $\dot{\boldsymbol{F}}$.

$$\dot{\boldsymbol{F}} = \frac{\boldsymbol{F}}{\Delta t} \quad \equiv \quad \dot{\boldsymbol{F}} = \frac{\partial}{\partial t}\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\right) \quad \longrightarrow \quad \dot{\boldsymbol{F}} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \quad \longrightarrow \quad \dot{\boldsymbol{F}} = \boldsymbol{l}\boldsymbol{F}$$

We can re-write the above thus:

$$\dot{\boldsymbol{F}} = \boldsymbol{l}\boldsymbol{F} \qquad \longrightarrow \qquad \boldsymbol{l} = \dot{\boldsymbol{F}}\,\boldsymbol{F}^{-1} \tag{9.12}$$

which suggests that the velocity gradient is the mapping function between the deformation gradient and the rate of change term of $\boldsymbol{F}$. It is acting on the current configuration of the material domain.

Just like the deformation gradient can be multiplicatively decomposed into its *stretch* and *rotation* tensors, the velocity gradient can likewise be decomposed additively according to Eq. 9.13:

$$\boldsymbol{l} = \boldsymbol{d}(\boldsymbol{x}, t) + \boldsymbol{w}(\boldsymbol{x}, t) \quad \longrightarrow \quad \boldsymbol{l} = \mathrm{sym}(\boldsymbol{l}) + \mathrm{asym}(\boldsymbol{l}), \tag{9.13}$$

where:

- $\boldsymbol{d}$ = ***rate of deformation tensor***, which is a *stretch-related symmetric* part of the velocity gradient. It is also commonly described as the *rate of strain tensor* or *strain rate*; and,
- $\boldsymbol{w}$ = ***continuum spin***, which is a *rotation-related antisymmetric/skew* part of the velocity gradient. It is also commonly called the *rate of rotation tensor* or *vorticity tensor*.

Mathematically, the $\boldsymbol{d}$ and $\boldsymbol{w}$ tensors can be written as:

$$\begin{aligned} &\text{Rate of deformation tensor:} \quad && \boldsymbol{d} = \frac{1}{2}\left(\boldsymbol{l} + \boldsymbol{l}^T\right) = \boldsymbol{d}^T \\ &\text{Continuum spin tensor:} \quad && \boldsymbol{w} = \frac{1}{2}\left(\boldsymbol{l} - \boldsymbol{l}^T\right) = -\boldsymbol{w}^T \end{aligned} \tag{9.14}$$

9.4 Measures of Strain

In this section, we present the different measures of strain used in computational mechanics. The discussion considers the classic expressions of normal and shear strain as usually presented in engineering mechanics before discussing the less common strain measures used in finite deformation literature such as the Cauchy-Green deformation tensors, and such like.

9.4.1 Normal and Shear Strain Measures

An important measure of material response is *strain*. Strain, in undergraduate textbooks, is defined as a measure of the relative change in size or shape of a body with respect to a reference configuration. For a material body shown in Fig. 9.8, we recall the following strain measures:

$$\text{Normal strain} \qquad \varepsilon_N = \frac{\Delta L}{L_0} \tag{9.15}$$

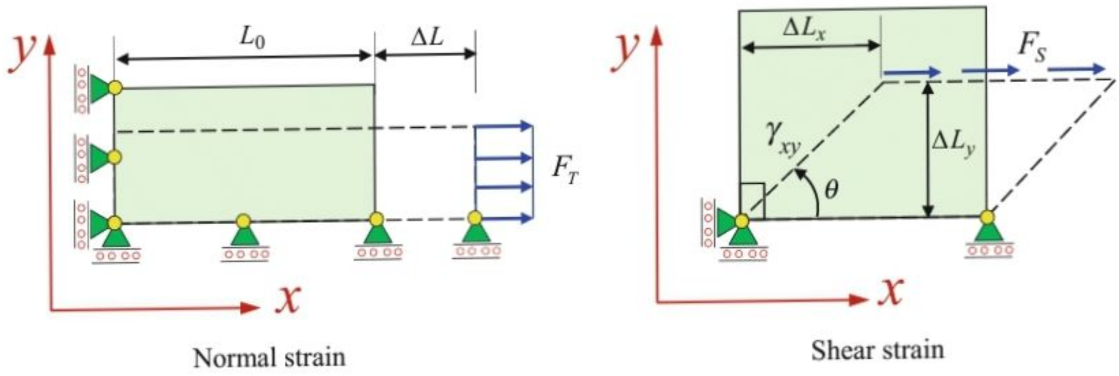

Fig. 9.8 Schematics of tensile and shear deformations

$$\text{Shear strain} \qquad \gamma_{xy} = \frac{\pi}{2} - \theta \quad \equiv \quad \tan^{-1}\left(\frac{\Delta L_x}{\Delta L_y}\right) \tag{9.16}$$

where L_0 is original length, and ΔL is the extension of the tensile test specimen. $\boldsymbol{F}_T$ and $\boldsymbol{F}_S$ are tensile and shear forces respectively. However, for a finite element process, the measures of strain adopted is based on the deformation gradient tensor derived previously. This way, not only small strain, but also finite strain, deformations can be accounted for within the FEM process.

9.4.2 Right Cauchy-Green Deformation Tensor

Consider again the material body of Fig. 9.5 and now let us define ds as the length of the line segment, $\overline{A'B'}$, in the deformed state. Therefore:

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} \qquad \longrightarrow \qquad \mathrm{d}s^2 &= (\boldsymbol{F}\mathrm{d}\boldsymbol{X}) \cdot (\boldsymbol{F}\mathrm{d}\boldsymbol{X}) \\ \mathrm{d}s^2 &= \mathrm{d}\boldsymbol{X}^T\boldsymbol{F}^T\boldsymbol{F}\mathrm{d}\boldsymbol{X} \\ \mathrm{d}s^2 &= \mathrm{d}\boldsymbol{X}^T\boldsymbol{C}\mathrm{d}\boldsymbol{X}. \end{aligned}$$

In the above, we defined $\boldsymbol{C}$, called the *right Cauchy-Green deformation tensor*, as:

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} \tag{9.17}$$

It is the first measure of strain that we can deduce based on the deformation gradient tensor. It is described as *right*, because the deformation gradient, $\boldsymbol{F}$, is on the right hand side of the transpose of $\boldsymbol{F}$ equation. It is also sometimes referred to as the *Green deformation tensor*. As shown above, the physical meaning of $\boldsymbol{C}$ is that it gives a measure of the square of the distances (ds^2) due to the deformation of a material body in the material reference frame.

9.4.3 *Left Cauchy-Green Deformation Tensor*

Similar to the approach taken above, we can also determine the length, $\mathrm{d}S$, of the material body in the spatial reference frame. Recall that in the spatial reference frame, $\mathrm{d}\boldsymbol{X} = \boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x}$. The distance $\mathrm{d}S$ can be determined according to the following:

$$\mathrm{d}S^2 = \mathrm{d}\boldsymbol{X}^T \cdot \mathrm{d}\boldsymbol{X} \qquad \longrightarrow \qquad \begin{aligned} \mathrm{d}S^2 &= \left(\boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x}\right)^T \left(\boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x}\right) \\ \mathrm{d}S^2 &= \mathrm{d}\boldsymbol{x}^T \left(\boldsymbol{F}^{-1}\right)^T \boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x} \\ \mathrm{d}S^2 &= \mathrm{d}\boldsymbol{x}^T \boldsymbol{B}^{-1}\mathrm{d}\boldsymbol{x} \end{aligned}$$

Notice that we have determined another type of strain measure, the *left Cauchy-Green deformation tensor*, which is represented by $\boldsymbol{B}$ thus:

$$\boldsymbol{B}^{-1} = \left(\boldsymbol{F}^{-1}\right)^T \boldsymbol{F}^{-1} \qquad \text{or simply} \qquad \boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T \tag{9.18}$$

The deformation gradient tensor, $\boldsymbol{F}$, is on the left hand side of the transpose of $\boldsymbol{F}$ hence the *left* Cauchy-green tensor. $\boldsymbol{B}$ is also called the *Finger deformation tensor*. Physically, the $\boldsymbol{B}$ tensor represents the square of the distances (dS^2) due to deformation of a material body in the spatial coordinates system.

9.4.4 *Change in Length, ΔL Measure*

Another common measure needed to determine the strain in a material body is the *change in length*. Having previously determined the distances in material (ds^2) and spatial (dS^2) reference frames, we will now go ahead and use them to determine the relative change in length for the material body.

Mathematically the change in length of a material body can be expressed as:

$$\Delta L^2 = \mathrm{d}s^2 - \mathrm{d}S^2 \qquad \longrightarrow \qquad \begin{aligned} \Delta L^2 &= \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{X} \cdot \mathrm{d}\boldsymbol{X} \\ \Delta L^2 &= \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{x} \cdot \boldsymbol{B}^{-1}\mathrm{d}\boldsymbol{x} \\ \Delta L^2 &= \mathrm{d}\boldsymbol{x} \cdot \left(\boldsymbol{I} - \boldsymbol{B}^{-1}\right)\mathrm{d}\boldsymbol{x}, \end{aligned}$$

where $\boldsymbol{I}$ is the identity tensor. The above demonstrates how the change in length i.e. *stretch* of a material body can be determined with respect to both material configurations. Under rigid body rotation, there is change in length ($\Delta L = 0$), therefore we note that:

$$\mathrm{d}\boldsymbol{x} \cdot \left(\boldsymbol{I} - \boldsymbol{B}^{-1}\right)\mathrm{d}\boldsymbol{x} = \boldsymbol{0} \qquad \longrightarrow \qquad \boldsymbol{B} = \boldsymbol{I}. \tag{9.19}$$

The above rigid-body motion condition implies that the mapping function between the material and spatial configuration (i.e. deformation gradient, $\boldsymbol{F}$) contains terms that do not cause a change in length of the material but rather a change in orientation.

9.4.5 Green Strain Tensor

It is vital that any strain measure used within a finite element scheme must return the same strain irrespective of orientation of the material body. This is a crucial requirement for formulation of strain in finite deformation systems. In particular, if the rotations arising from deformation of a material body grows, so does the errors arising in the strain formulation. It is a requirement therefore that an objective strain measure must not be polluted by rigid body rotations [3]. Hereafter, we introduce the next measure of strain which obeys these requirements.

Consider the material body of Fig. 9.5, we can derive the change of length in terms of material reference frame thus:

$$\begin{aligned}
\Delta L^2 = \mathrm{d}s^2 - \mathrm{d}S^2 \quad &\longrightarrow \quad \Delta L^2 = \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{X} \cdot \mathrm{d}\boldsymbol{X} \\
&\longrightarrow \quad \Delta L^2 = (\boldsymbol{F}\mathrm{d}\boldsymbol{X}) \cdot (\boldsymbol{F}\mathrm{d}\boldsymbol{X}) - \mathrm{d}\boldsymbol{X} \cdot \mathrm{d}\boldsymbol{X} \\
&\qquad\quad \Delta L^2 = \mathrm{d}\boldsymbol{X} \cdot \left(\boldsymbol{F}^T\boldsymbol{F}\right) \mathrm{d}\boldsymbol{X} - \mathrm{d}\boldsymbol{X} \cdot (\boldsymbol{I})\mathrm{d}\boldsymbol{X} \\
&\qquad\quad \Delta L^2 = \mathrm{d}\boldsymbol{X} \cdot \left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) \mathrm{d}\boldsymbol{X}
\end{aligned}$$

In the above, we notice that we have defined a new term: $\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}$, which we will now equate to $2\boldsymbol{E}$ such that:

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) \quad \longrightarrow \quad \boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{C} - \boldsymbol{I}\right) \tag{9.20}$$

where the new term, $\boldsymbol{E}$, is called the *Green-Lagrange strain tensor* or *Green strain tensor* for short. Based on Eq. 9.20, we can observe that:

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) \quad \longrightarrow \quad \boldsymbol{E} = \frac{1}{2}\frac{\Delta L^2}{\mathrm{d}\boldsymbol{X} \cdot \mathrm{d}\boldsymbol{X}} \tag{9.21}$$

Notice that Eq. 9.21 is similar to the engineering *normal* strain measure of Eq. 9.15 being a ratio of a change in length (ΔL^2) to the original length, $L_0^2 \equiv |\mathrm{d}\boldsymbol{X} \cdot \mathrm{d}\boldsymbol{X}|$, in the material reference frame. It is commonly regarded in continuum mechanics that this expression of normal strain (Eq. 9.20) is a more accurate measure of strain than that of Eq. 9.15.

Since any error due to rigid body rotation must be removed for a finite element strain measure, we will now assess if the strain, $\boldsymbol{E}$, is independent of rigid body motion. Recall the polar decomposition of the deformation gradient, $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$, thus we will now substitute these into Eq. 9.20.

$$\begin{aligned} \boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} \quad \longrightarrow \quad \boldsymbol{C} &= (\boldsymbol{R}\boldsymbol{U})^T\,\boldsymbol{R}\boldsymbol{U} = \boldsymbol{U}^T\left(\boldsymbol{R}^T\boldsymbol{R}\right)\boldsymbol{U} \\ &= \boldsymbol{U}^T\,(\boldsymbol{I})\,\boldsymbol{U} = \boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{U}^2 \end{aligned} \tag{9.22}$$

Since $\boldsymbol{R}$ is orthogonal (i.e. $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$) and $\boldsymbol{U}$ is symmetric (i.e. $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{U}^2$), it can be seen that:

$$\boldsymbol{E} = \frac{1}{2}\,(\boldsymbol{C} - \boldsymbol{I}) \quad \longrightarrow \quad \boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{U}^2 - \boldsymbol{I}\right) \tag{9.23}$$

which establishes that the Green-Lagrange strain tensor is independent of rigid body rotation as it is solely a function of the stretch tensor. It gives a reliable strain measure for use in describing the material response within a finite element modelling scheme.

9.4.6 *Almansi Strain Tensor*

Similar to the Green-Lagrange strain tensor, we will also introduce another strain measure whose definition is independent of the rigid body rotation associated with a material body subjected to a deformation gradient, $\boldsymbol{F}$. Consider again the material body of Fig. 9.5, we can derive the change of length in terms of spatial reference frame thus:

$$\begin{aligned} \Delta L^2 = \mathrm{d}s^2 - \mathrm{d}S^2 \quad \longrightarrow \quad \Delta L^2 &= \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{X}\cdot\mathrm{d}\boldsymbol{X} \\ \Delta L^2 &= \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} - \left(\boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x}\right)\cdot\left(\boldsymbol{F}^{-1}\mathrm{d}\boldsymbol{x}\right) \\ \Delta L^2 &= \mathrm{d}\boldsymbol{x}\cdot(\boldsymbol{I})\mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{x}\cdot\left(\boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\right)\mathrm{d}\boldsymbol{x} \\ \Delta L^2 &= \mathrm{d}\boldsymbol{x}\cdot\left(\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\right)\mathrm{d}\boldsymbol{x} \end{aligned}$$

We have now introduced another alternative strain measure called the *Almansi strain tensor*, $\boldsymbol{e}$ such that:

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\right) \quad \longrightarrow \quad \boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{B}^{-1}\right) \tag{9.24}$$

This equation shares a similar definition as to the Green-Lagrange strain tensor except that the derivation is based on spatial material reference frame. It is the strain measure of a material body that is being subjected to a deformation gradient with the observer sat at a static location and observing the deformation of the body.

We will also assess if this strain is not affected by rigid body rotation. Again, recall the polar decomposition of $\boldsymbol{F}$ in the spatial reference frame thus: $\boldsymbol{F} = \boldsymbol{VR}$. If we substitute this into Eq. 9.24, we will obtain:

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{B}^{-1}\right) \quad \longrightarrow \quad \begin{aligned} \boldsymbol{e} &= \frac{1}{2}\left\{\boldsymbol{I} - \left[\boldsymbol{F}\boldsymbol{F}^{T}\right]^{-1}\right\} = \frac{1}{2}\left\{\boldsymbol{I} - \left[(\boldsymbol{VR})(\boldsymbol{VR})^{T}\right]^{-1}\right\} \\ &= \frac{1}{2}\left\{\boldsymbol{I} - \left[\boldsymbol{V}(\boldsymbol{R}^{T}\boldsymbol{R})\boldsymbol{V}^{T}\right]^{-1}\right\} \\ &= \frac{1}{2}\left\{\boldsymbol{I} - \left[\boldsymbol{V}\boldsymbol{V}^{T}\right]^{-1}\right\} = \frac{1}{2}\left\{\boldsymbol{I} - \left[\boldsymbol{V}^{2}\right]^{-1}\right\} \end{aligned}$$

which establishes that the Almansi strain tensor is solely a function of the left stretch tensor, $\boldsymbol{V}$, and so not affected by any rigid body rotation of the material body under the effect of $\boldsymbol{F}$. We can also observe that:

$$\boldsymbol{B}^{-1} = \left[\boldsymbol{V}^{2}\right]^{-1} \quad \longrightarrow \quad \boldsymbol{B}^{-1} = \left[\boldsymbol{V}^{-1}\right]^{2} \tag{9.25}$$

9.4.7 Logarithmic Strain Tensor

Another commonly used strain measure, especially in reporting experimental work is the *logarithmic strain*. Let us consider an infinitesimally small line element of initial length, L_0, in its undeformed configuration. If it is subjected to a deformation gradient, $\boldsymbol{F}$, resulting in a final deformed length, l, having experienced a change in length, ΔL, we can define the *stretch* or *extension ratio*, λ thus:

$$\lambda = \frac{l}{L_0} \quad \longrightarrow \quad \lambda = \frac{L_0 + \Delta L}{L_0} = 1 + \frac{\Delta L}{L_o} = 1 + \varepsilon_N \tag{9.26}$$

where ε_N is *normal engineering strain* defined in Eq. 9.15. The above definition considers strain changes to be uniform over the line segment. However, it is common in plasticity for any strain increment to be nonlinear, in which case, the path of strain increment becomes important. We use the *logarithmic strain* to capture this type of strain increment.

If the infinitesimally small line element experiences a local change in length, δl, the resulting strain increment with respect to the final length, l, can be expressed:

$$\delta\varepsilon = \frac{\delta l}{l} \tag{9.27}$$

The sum of all strain-path-dependent strain increments becomes the logarithmic strain, which is the integration of the strain increments from the initial length, L_0, to the final length, l, thus:

$$\int \delta\varepsilon = \int_{L_0}^{l} \frac{1}{l}\delta l \qquad \longrightarrow \qquad \begin{aligned} \varepsilon &= \ln\left(\frac{l}{L_0}\right) \\ &= \ln(\lambda) \\ &= \ln(1+\varepsilon_N) \\ &= \varepsilon_N - \frac{1}{2}(\varepsilon_N)^2 + \frac{1}{3}(\varepsilon_N)^3 - \cdots \end{aligned} \tag{9.28}$$

For a three-dimensional material body, we can define the same logarithmic strain in terms of the deformation gradient, $\boldsymbol{F}$, thus:

$$\boldsymbol{\varepsilon} = -\frac{1}{2}\ln \boldsymbol{B}^{-1} \qquad \longrightarrow \qquad \boldsymbol{\varepsilon} = -\frac{1}{2}\ln\left(\boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\right) \tag{9.29}$$

The logarithmic strain tensor is also referred to as the *Hencky strain tensor*, named after H Hencky who, in 1928, made this derivation. It is necessary for finite element modelling studies that the strain tensor must not be affected by rigid body rotations, we will now show that the logarithmic strain tensor obeys this requirement.

We established previously in Eq. 9.25 that $\boldsymbol{B}$ can be related to the left stretch tensor, $\boldsymbol{V}$. We will exploit this relationship to re-define the logarithmic strain tensor thus:

$$\boldsymbol{\varepsilon} = -\frac{1}{2}\ln \boldsymbol{B}^{-1} \qquad \longrightarrow \qquad \boldsymbol{\varepsilon} = -\frac{1}{2}\ln\left[\boldsymbol{V}^{-1}\right]^2 = \ln \boldsymbol{V} \tag{9.30}$$

which shows that the logarithmic strain tensor is independent of the rotation tensor. It is thus a suitable measure of strain for finite element modelling studies.

9.4.8 Seth-Hill Family of Strain Tensors

Although previously, we have presented the different measures of strain, they can be combined into a single equation, which is generally regarded as the *Seth-Hill family of strain tensors*. For a material body, $\Omega \subset \Re^n$, subjected to a deformation gradient, $\boldsymbol{F}$, which can be decomposed into $\boldsymbol{F} = \boldsymbol{RU} = \boldsymbol{VR}$, we can define the Seth-Hill family of strains in terms of either the right stretch tensor, $\boldsymbol{U}$, or the left stretch tensor, $\boldsymbol{V}$, as:

$$\boldsymbol{\varepsilon} = \frac{1}{n}\left(\boldsymbol{U}^n - \boldsymbol{I}\right) \qquad \text{or} \qquad \boldsymbol{\varepsilon} = \frac{1}{n}\left(\boldsymbol{V}^n - \boldsymbol{I}\right) \tag{9.31}$$

where n is a real number and not necessarily an integer. The above equation is also referred to as the *generalized strain measures in Lagrangian and Eulerian descriptions*. For specific values of n, Eq. 9.31 shows any of the earlier strain measures as illustrated below.

$$
\begin{array}{lll}
\text{For } n = -2: & \text{Euler-Almansi strain,} & \boldsymbol{\varepsilon} = 0.50\left(\boldsymbol{I} - \boldsymbol{V}^{-2}\right) \\
\text{For } n = -1: & \text{“True” strain,} & \boldsymbol{\varepsilon} = \boldsymbol{I} - \ln \boldsymbol{U}^{-1} \\
\text{For } n \to 0: & \text{Logarithmic strain,} & \boldsymbol{\varepsilon} = \ln \boldsymbol{U} \text{ or } \ln \boldsymbol{V} \\
\text{For } n = 1: & \text{Engineering strain,} & \boldsymbol{\varepsilon} = \boldsymbol{U} - \boldsymbol{I} \text{ or } \boldsymbol{V} - \boldsymbol{I} \\
\text{For } n = 2: & \text{Green-Lagrange strain,} & \boldsymbol{\varepsilon} = 0.50\left(\boldsymbol{U}^2 - \boldsymbol{I}\right)
\end{array}
$$

The Seth-Hill family of strains can be demonstrated for the elongation of a cylindrical bar of cross-sectional area, A, original length, L_0, undergoing a tensile deformation, ΔL, such that the final length, $L = L_0 + \Delta L$. The resulting stretch ratio becomes: $\lambda = L/L_0$.

The stretch ratio, λ, will always be positive, $\lambda > 0$ since lengths will always be positive. Therefore, the expressions of different strain measures, adapted from the Seth-Hill generalized family of strains, are given below while the comparison plots of the strain measures are shown in Fig. 9.9.

$$
\begin{array}{lllll}
\text{For } n = -2: & \text{Euler-Almansi strain,} & \varepsilon = & \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right) & = \frac{1}{2}\left(\frac{L^2 - L_0^2}{L^2}\right) \\
\text{For } n = -1: & \text{“True” strain,} & \varepsilon = & 1 - \frac{1}{\lambda} & = \frac{L - L_0}{L} \\
\text{For } n \to 0: & \text{Logarithmic strain,} & \varepsilon = & \ln \lambda & = \ln\left(\frac{L}{L_0}\right) \\
\text{For } n = 1: & \text{Engineering strain,} & \varepsilon = & \lambda - 1 & = \frac{L - L_0}{L_0} \\
\text{For } n = 2: & \text{Green-Lagrange strain,} & \varepsilon = & \frac{1}{2}\left(\lambda^2 - 1\right) & = \frac{1}{2}\left(\frac{L^2 - L_0^2}{L_0^2}\right)
\end{array}
\tag{9.32}
$$

Example 9.1 A 3×2 mm^2 rectangular plate shown in Fig. 9.10 is subjected to a uniaxial tensile deformation such that the plate undergoes a stretch of 50% along the x–axis and a contraction of 20% along the y–axis. The material configuration of the deformed the plate is given by:

$$
\begin{aligned}
x &= 1.5X + 0.0Y \\
y &= 0.0X + 0.8Y
\end{aligned}
\tag{9.33}
$$

(a) Determine the deformation gradient, $\boldsymbol{F}$, for the above problem.
(b) Determine the stretch, $\boldsymbol{U}$ and rotation, $\boldsymbol{R}$, tensors.
(c) Determine the Green-Lagrange strain tensor, $\boldsymbol{E}$.

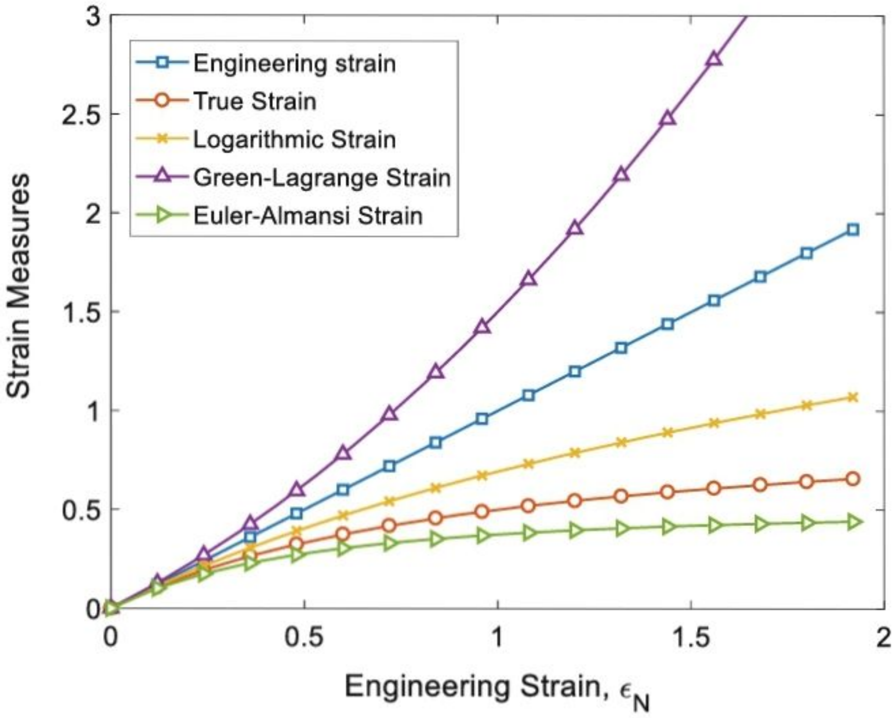

Fig. 9.9 A comparison of all the strain measures for uniaxial deformation of a rod

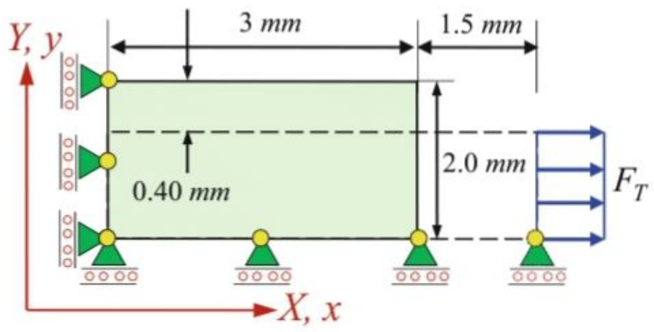

Fig. 9.10 Uniaxial deformation of a rectangular plate

(d) Determine the Engineering strain, $\boldsymbol{\varepsilon}_{engineering}$, and prove that this yields the expected strains.

Solution

(a) **Deformation gradient tensor:** Based on Eq. 9.33, we get:

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \quad \longrightarrow \quad \boldsymbol{F} = \begin{bmatrix} \dfrac{\partial x}{\partial X} & \dfrac{\partial x}{\partial Y} \\[2ex] \dfrac{\partial y}{\partial X} & \dfrac{\partial y}{\partial Y} \end{bmatrix} = \begin{bmatrix} 1.5 & 0.0 \\ 0.0 & 0.8 \end{bmatrix} \tag{9.34}$$

(b) **Stretch tensors**: We established previously that the stretch, $\boldsymbol{U}$, can be related to the left Cauchy-Green tensor, $\boldsymbol{C}$. We will use this to determine the stretch tensor only as given below:

$$C = F^T F = U^2 \quad \longrightarrow \quad U = \sqrt{F^T F} \tag{9.35}$$

Substituting Eq. 9.34 into Eq. 9.35 becomes:

$$U = \sqrt{F^T F} \quad \longrightarrow \quad U = \sqrt{F^2} = F \quad \text{since} \quad F^T = F$$

$$U = \begin{bmatrix} 1.5 & 0.0 \\ 0.0 & 0.8 \end{bmatrix} = \begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix}$$

The above result shows the deformation is dominated by stretch along the $X-$ and $Y-$directions, hence the diagonal terms: $U_{xx} = \lambda_{xx} = 4.5/3.0 = 1.5$ and $U_{yy} = \lambda_{yy} = 1.6/2.0 = 0.8$, where $\lambda = L/L_0$ is the stretch ratio.

(c) **Rotation tensors**: Similar to the derivation of the stretch tensor, we will also calculate the rotation tensor based on F and U. Recall,

$$F = RU \quad \longrightarrow \quad R = \frac{F}{U} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

Assuming the angle of rotation is φ, we also note that:

$$R = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \quad \longrightarrow \quad R = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \tag{9.36}$$

The above implies that: $\cos\varphi = 1.0 \quad \longrightarrow \quad \varphi = \cos^{-1} 1.0 = 0^0$. Therefore, in the deformed configurations, the edges have not been rotated, hence rigid body rotation is excluded, as expected for a uniaxial test.

(d) **Green-Lagrange strain tensor**: This is determined as follows:

$$E = \frac{1}{2}\left(F^T F - I\right) \quad \longrightarrow \quad E = \begin{bmatrix} 0.625 & 0.000 \\ 0.000 & -0.180 \end{bmatrix} \tag{9.37}$$

(e) **Engineering strain tensor**: This is determined using Eq. 9.32 thus:

$$\varepsilon_{engineering} = U - I \quad \longrightarrow \quad \varepsilon_{engineering} = \begin{bmatrix} 1.5 & 0.0 \\ 0.0 & 0.8 \end{bmatrix} - \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

$$\text{Therefore,} \quad \varepsilon_{engineering} = \begin{bmatrix} 0.5 & 0.0 \\ 0.0 & -0.2 \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \tag{9.38}$$

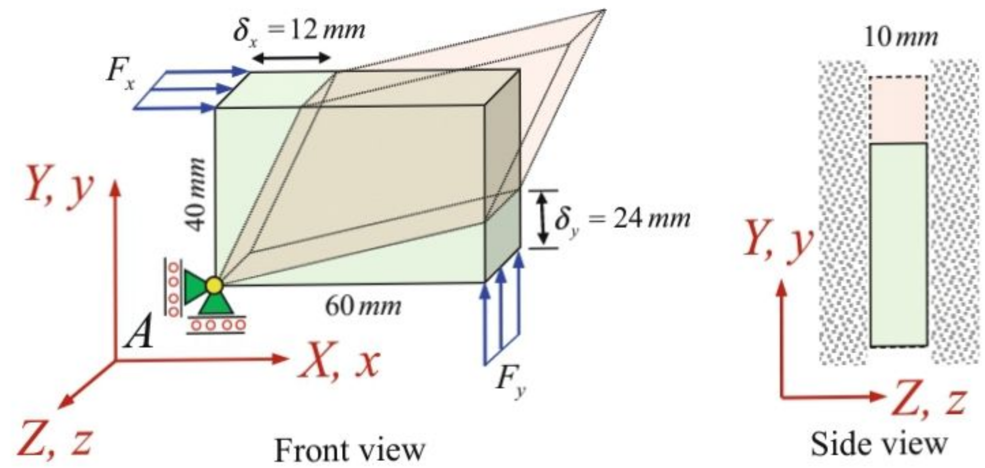

Fig. 9.11 Deformation of a PTFE plate subjected to two shear forces

Notice that along the x-axis, $\varepsilon_{xx} = 0.5$, which is a 50% increase in the $X-$edge and $\varepsilon_{yy} = -0.2$ is a 20% contraction of the $Y-$edge. This is the same load imposed on the plate thus demonstrating the calculations are correct.

Example 9.2 A PTFE sheet of dimensions $60 \times 40 \times 10$ mm^3 shown in Fig. 9.11 is pinned securely at A and constrained along the $z-$axis. The plate is subjected to two shear forces, F_x and F_y, leading to edge deformations of $\delta_x = 12$ mm and $\delta_y = 24$ mm respectively. Assume that the deformation in the through-thickness direction ($z-$axis) is minimal.

(a) Determine the mapping function that relates material points in the material reference frame to those of the spatial reference frame.
(b) Determine the deformation gradient, $\boldsymbol{F}$, for the above problem.
(c) Determine the stretch, $\boldsymbol{U}$ and $\boldsymbol{V}$, and rotation, $\boldsymbol{R}$, tensors.
(d) Determine the Euler-Almansi strain tensor, $\boldsymbol{e}$.

Solution

(a) **Mapping function in spatial reference frame:** We develop the relationship between material and spatial configurations. Let us identify the material reference frame as: $XYZ-$axes and the spatial reference frame as: $xyz-$axes. Consider that the origin of both coordinate systems coincide as shown in Fig. 9.11. The mapping function along the $x-$axis should take the form:

$$x = \chi_x X + \chi_y Y + +\chi_z Z \tag{9.39}$$

where χ_x, χ_y and χ_z are the *coefficients/weighting terms* that describe the axis-specific mapping functions for $X-$, $Y-$ and $Z-$axes respectively. We will now attempt to define what these coefficient/weighting terms are.

Recall that the deformation gradient tensor, $\boldsymbol{F}$, that caused the observed deformation is defined as:

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \quad \longrightarrow \quad \boldsymbol{F} = \begin{bmatrix} \dfrac{\partial x}{\partial X} & \dfrac{\partial x}{\partial Y} & \dfrac{\partial x}{\partial Z} \\ \dfrac{\partial y}{\partial X} & \dfrac{\partial y}{\partial Y} & \dfrac{\partial y}{\partial Z} \\ \dfrac{\partial z}{\partial X} & \dfrac{\partial z}{\partial Y} & \dfrac{\partial z}{\partial Z} \end{bmatrix} \tag{9.40}$$

Considering the first row of the matrix of Eq. 9.39 and the coefficients of Eq. 9.40, we can conclude that for the $x-$axis:

$$\chi_x = \frac{\partial x}{\partial X} \qquad \chi_y = \frac{\partial y}{\partial X} \quad \text{and} \quad \chi_z = \frac{\partial z}{\partial X} \tag{9.41}$$

This implies that the first weighting term that describes the mapping function (i.e. χ_x) is essentially the ratio of the lengths: $\mathrm{d}\boldsymbol{x}$ and $\mathrm{d}\boldsymbol{X}$. Based on Fig. 9.11, along the $x-$axis, we notice that:

$$\chi_x = \frac{\partial x}{\partial X} \quad \equiv \quad \boldsymbol{\chi}_x = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\boldsymbol{X}}$$

For the PTFE plate under shear loading, along the $x-$axis, material and spatial positions are coincident hence $\mathrm{d}x = 40$ and $\mathrm{d}X = 40$, so $\chi_x = \mathrm{d}x/\mathrm{d}X = 40/40 = 1.0$. Also, in the $y-$axis, $\chi_y = \dfrac{\partial x}{\partial Y} \equiv \chi_y = \dfrac{\mathrm{d}x}{\mathrm{d}Y} = \dfrac{12\text{ mm}}{60\text{ mm}} = 0.40$. There is no deformation in the $z-$axis, hence: $\mathrm{d}z = \mathrm{d}Z$ (i.e. material and spatial reference frames are coincident).

Using this approach, we can determine all the weighting terms and the resulting mapping functions for the problem are:

$$x = \left(\frac{\mathrm{d}x}{\mathrm{d}X}\right)X + \left(\frac{\mathrm{d}x}{\mathrm{d}Y}\right)Y + \left(\frac{\mathrm{d}x}{\mathrm{d}Z}\right)Z \quad \longrightarrow \quad x = 1.0X + 0.4Y + 0.0Z$$

$$y = \left(\frac{\mathrm{d}y}{\mathrm{d}X}\right)X + \left(\frac{\mathrm{d}y}{\mathrm{d}Y}\right)Y + \left(\frac{\mathrm{d}y}{\mathrm{d}Z}\right)Z \quad \longrightarrow \quad y = 0.3X + 1.0Y + 0.0Z$$

$$z = \left(\frac{\mathrm{d}z}{\mathrm{d}X}\right)X + \left(\frac{\mathrm{d}z}{\mathrm{d}Y}\right)Y + \left(\frac{\mathrm{d}z}{\mathrm{d}Z}\right)Z \quad \longrightarrow \quad z = 0.0X + 0.0Y + 1.0Z$$

(b) **Deformation gradient tensor:** Based on the mapping functions and Eq. 9.40, $\boldsymbol{F}$ becomes:

$$\boldsymbol{F} = \begin{bmatrix} 1.0 & 0.4 & 0.0 \\ 0.3 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \tag{9.42}$$

(c) **Stretch and rotation tensors:** The right stretch tensor, $\boldsymbol{U}$ is determined using the deformation gradient tensor.

$$\boldsymbol{U} = \sqrt{\boldsymbol{F}^T\boldsymbol{F}} \quad \longrightarrow \quad \boldsymbol{U} = \begin{bmatrix} 1.0440 & 0.8367 & 0.0000 \\ 0.8367 & 1.0770 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

Similarly, the left stretch tensor, $\boldsymbol{V}$ is given below.

$$\boldsymbol{V} = \sqrt{\boldsymbol{F}\boldsymbol{F}^T} \quad \longrightarrow \quad \boldsymbol{V} = \begin{bmatrix} 1.0770 & 0.8367 & 0.0000 \\ 0.8367 & 1.0440 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

The are two possible rotation tensors. Here, $\boldsymbol{R}_1$ can be determined from $\boldsymbol{U}$ while $\boldsymbol{R}_2$ is determined using $\boldsymbol{V}$ thus:

$$\boldsymbol{R}_1 = \boldsymbol{F}\boldsymbol{U}^{-1} \quad \longrightarrow \quad \boldsymbol{R}_1 = \begin{bmatrix} 1.7490 & -0.9873 & 0.0000 \\ -1.2099 & 1.8684 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

and

$$\boldsymbol{R}_2 = \boldsymbol{V}^{-1}\boldsymbol{F} \quad \longrightarrow \quad \boldsymbol{R}_2 = \begin{bmatrix} 1.8684 & -0.9873 & 0.0000 \\ -1.2099 & 1.7490 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

(d) **Euler-Almansi strain tensor:** This is derived using the left stretch tensor, $\boldsymbol{V}$:

$$\boldsymbol{e} = \frac{1}{2}\left\{\boldsymbol{I} - (\boldsymbol{V}^2)^{-1}\right\} \quad \longrightarrow \quad \boldsymbol{e} = \begin{bmatrix} -4.4677 & 4.9250 & 0.0000 \\ 4.9250 & -4.6620 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

9.5 Measures of Stress

The previous section introduced the kinematics of motion and deformation of a material body when subjected to a deforming stimulus. The deformation experienced can be quantified using the measures introduced previously. The consequence of a deforming body is the notion of *stress*. It can be also simply described as the *force per unit area.* The concept of internal forces is central to the discussion of stresses. This section discusses the formulation of *traction vectors, stress tensors* and diverse *stress measures* used in describing the state of stress of a material body undergoing finite deformation.

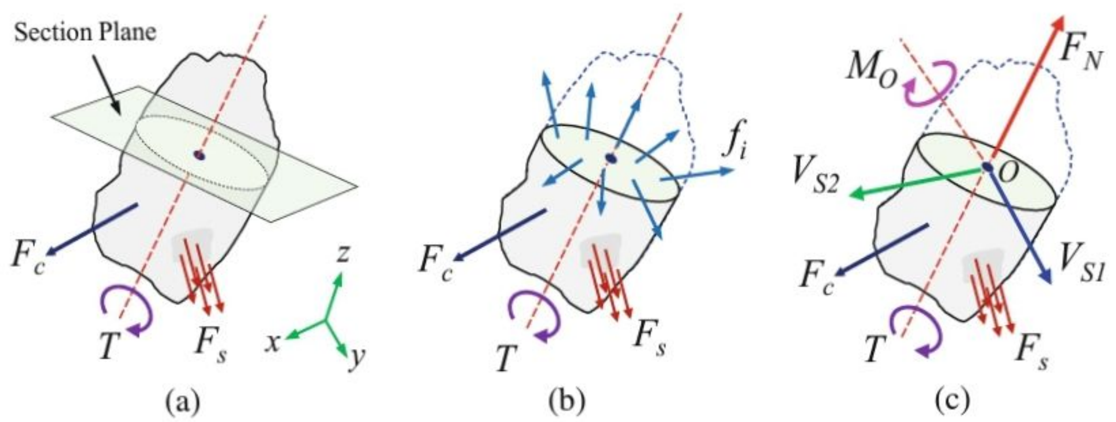

Fig. 9.12 An illustration of action of internal forces: (**a**) a material body under effect of external loads; (**b**) resulting internal forces acting on section plane; and, (**c**) resultant internal forces under static equilibrium conditions

9.5.1 The Concept of Internal Forces

In this section, we introduce the concept of *internal forces*, highlighting how crucial they are in describing the stress acting within a material body. Consider the material body shown in Fig. 9.12a under the effect of three *external loads* namely: concentrated force, F_c, surface forces, F_s and torque, T, acting about the longitudinal $z-$axis of the body.

The effect of the external forces is to excite an internal material response within the material body. To visualize this internal response, a *section plane* is introduced to 'cut through' the material body at a pre-defined location with respect to a chosen xyz reference frame.

Figure 9.12b shows the sectioned view of the material body revealing the set of internal forces, f_i where $i = 1, 2, \cdots, \infty$. On a given sectioned face, there can be very large numbers of internal forces, f_i, which result from the effect of the sectioned-off part on the remaining part of the material body. The size and orientation of these internal forces depend on the cumulative effect of the external loads. Material response derives from the effect of these internal forces.

If a typical internal force, f_i, is oriented with respect to the $x-$, $y-$and $z-$axes by the angles α_i, β_i and γ_i respectively, then the force vector system for this internal force becomes:

$$\boldsymbol{f}_i = f\cos\alpha_i\boldsymbol{i} + f\cos\beta_i\boldsymbol{j} + f\cos\gamma_i\boldsymbol{k} \quad \text{where} \quad f = \text{ magnitude of } f_i \qquad (9.43)$$

The description of the material response will rely on a comprehensive analysis of the complete 3D set of force systems that are active on the sectioned surface. It is not always analytically convenient to determine this set of force systems hence it is common in continuum mechanics to make some assumptions about the relationship between the external loads and the complete set of internal force systems. The simplest assumption is that the set of internal forces must be in *static equilibrium* with the set of external forces. This requires that equations of static equilibrium are applied on the set on internal forces to reduce their effect to a few forces.

As shown in Fig. 9.12c, the resulting set of internal forces following application of equations of static equilibrium include: *axial normal force*, $\boldsymbol{F}_N$, two *shear forces*, $\boldsymbol{V}_{S1}$ and $\boldsymbol{V}_{S2}$, and *moment*, $\boldsymbol{M}_O$, about point O. Mathematically:

$$\begin{aligned}
&\text{Sum of forces along the z-axis, Axial force:} && \sum_{i=1}^{\infty} \boldsymbol{f}_i &&= \boldsymbol{F}_N \\
&\text{Sum of forces along y-axis, shear force:} && \sum_{i=1}^{\infty} \boldsymbol{f}_i &&= \boldsymbol{V}_{S1} \\
&\text{Sum of forces along the x-axis, shear force:} && \sum_{i=1}^{\infty} \boldsymbol{f}_i &&= \boldsymbol{V}_{S2} \\
&\text{Sum of moment with respect to point O, moment :} && \sum_{i=1}^{\infty} \boldsymbol{r}_i \otimes \boldsymbol{f}_i &&= \boldsymbol{M}_O
\end{aligned} \tag{9.44}$$

where $\boldsymbol{r}_i$ is the position vector of each internal force, $\boldsymbol{f}_i$. All the terms in Eq. 9.44 are internal quantities since they were derived by the internal force vector, $\boldsymbol{f}_i$. These internal forces are the basis upon which stresses, strains, and other continuum mechanics parameters associated with material response are derived.

> *Stress is the measure of the intensity of an internal force per unit area where the force acts*. In other words, it is the ratio of the internal force at a point to the area over which that internal force is acting. High stress over the same unit implies high internal force and vice versa.

9.5.2 *The Cauchy Stress Tensor*

Let us now consider the concept of stress tensors and how these relate to the internal forces. Consider the material body shown in Fig. 9.13, but subject to the same external loads as Fig. 9.12a. Across the sectioned surface, there exist many internal force system, $\boldsymbol{f}$. We first isolate a reference configuration with an *infinitesimal surface spatial element*, dS, located at position, $\boldsymbol{X}$, having a surface traction, $\boldsymbol{T}$. This surface element has an outward normal, $\boldsymbol{N}$ which coincides with the unit vector at $\boldsymbol{X}$. If we impose the equations of static equilibrium on the surface element, we will obtain an infinitesimal *resultant internal force*, d$\boldsymbol{f}$ as shown in Fig. 9.13.

Similarly, the material body of Fig. 9.13 can be isolated in a *current configuration* after the *reference configuration* material points have been transformed using a time-dependent mapping function, $M(t)$. We can identify the current configuration surface element, ds which is located at $\boldsymbol{x}$ such that the resultant force, d$\boldsymbol{f}$ exists with an outward normal vector, $\boldsymbol{n}$, and surface traction vector, $\boldsymbol{t}$. The infinitesimal resultant force can be defined analytically thus:

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{T}\mathrm{d}S \quad \text{or} \quad \mathrm{d}\boldsymbol{f} = \boldsymbol{t}\mathrm{d}s \tag{9.45}$$

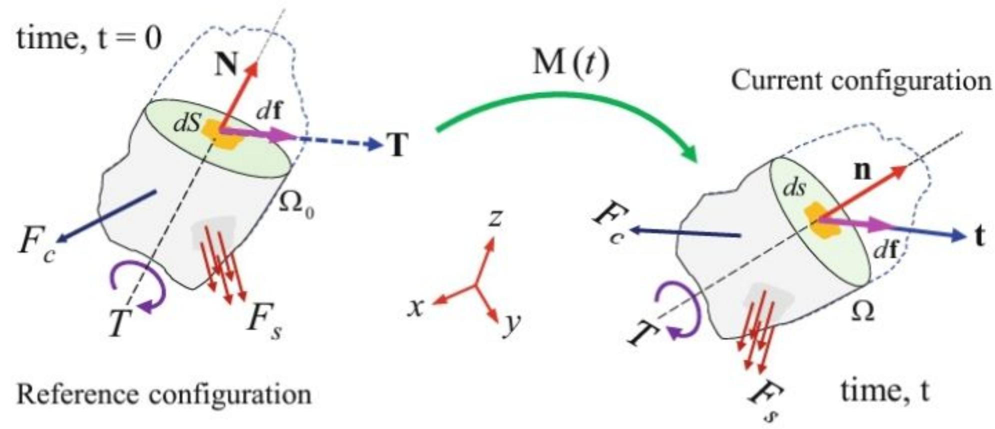

Fig. 9.13 An illustration of surface traction vectors on a material body

The *surface traction* ($\boldsymbol{T}$ and $\boldsymbol{t}$) is the *force per unit surface area*. In the context of Eq. 9.45, $\boldsymbol{t}$ is called the *Cauchy traction vector*, which is the force per unit surface area, and ds defined on the current configuration. It is also regarded as the *true traction vector*.

On the other hand, the surface traction, $\boldsymbol{T}$, is called the *first Piola-Kirchoff traction vector*. It is the force per unit surface area, dS defined in the reference configuration. It is also called the *nominal traction vector* and is often described as a *pseudo traction vector* [5]. It is a virtual traction vector acting in the reference configuration, but with orientation coincident with the Cauchy traction vector, hence its representation by dash lines in Fig. 9.13. Examples of surface traction vectors include contact forces between contacting surfaces, and friction vectors, as well as hydrostatic forces. Even the kinetics of wind speed on a turbine blade constitute an example of a type of surface force.

According to the definitions of the surface traction vectors, i.e. force per unit surface area defined in a given configuration, one can make the projection that there must exist a relationship between surface traction vectors and a stress tensor since both quantities share the same unit. This extrapolation was made by Augustine-Louis Cauchy in what is now known as the *Cauchy stress theorem*.

The Cauchy stress theorem postulates that there must exist a second-order tensor field, $\boldsymbol{\sigma}$, in the current configuration or $\boldsymbol{P}$, in the reference configuration, such that:

$$\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) = \boldsymbol{\sigma}(\boldsymbol{x}, t)\boldsymbol{n} \qquad \text{or} \qquad \boldsymbol{T}(\boldsymbol{X}, t, \boldsymbol{N}) = \boldsymbol{P}(\boldsymbol{X}, t)\boldsymbol{N} \tag{9.46}$$

where $\boldsymbol{\sigma}$ is a *symmetric* spatial tensor field called the *Cauchy stress tensor*. It is also simply called the *Cauchy stress* or more appropriately, in line with undergraduate engineering mechanics, the true stress tensor. Also, $\boldsymbol{P}$ is called the *first Piola-Kirchoff stress tensor*. It is also regarded simply as the *Piola stress* or the *nominal stress tensor*.

In matrix notation, we re-write the current configuration representation of Eq. 9.46, for the $xyz-$reference frame, thus:

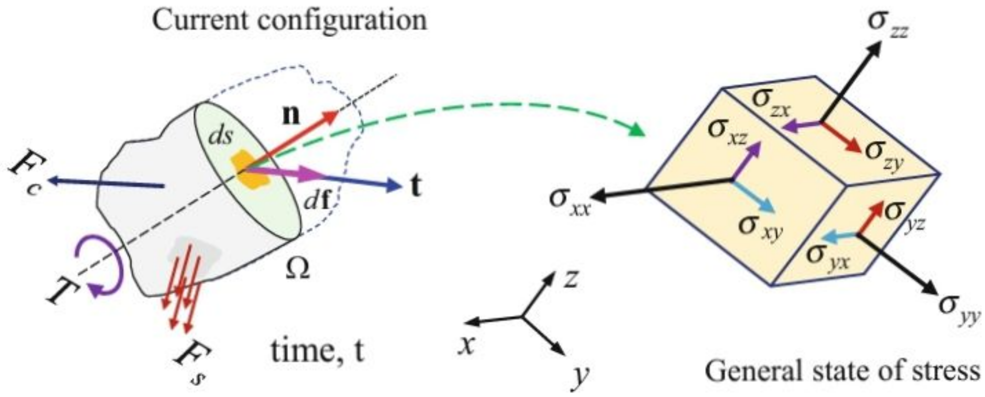

Fig. 9.14 The stress components that define the general state of stress within a material body

$$[t] = [\sigma]\,[n] \qquad \longrightarrow \qquad \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \tag{9.47}$$

In the above, $[\sigma]$ is the *Cauchy stress matrix* and it is a 3×3 matrix where the diagonal terms represents the normal stress terms and the off-diagonal terms are measures of shear stresses. The Cauchy stress matrix gives a comprehensive representation of a general state of stress for a point within a material body, as shown in Fig. 9.14.

As the material body continues to experience finite deformation, the material point of interest will change with position, time and orientation (through the outward normal vector). The Cauchy stress matrix must obey the principles of stress transformation discussed in undergraduate mechanics textbooks. The graphical representation of these orientation-dependent changes in magnitude of terms of the Cauchy stress matrix, is the *Mohr's circle of stress*.

Due to the requirement for static equilibrium and conservation of angular moment to be obeyed for a material body undergoing small deformations, we conclude that: $\sigma_{xy} \equiv \sigma_{yx}$, $\sigma_{xz} \equiv \sigma_{zx}$ and $\sigma_{yz} \equiv \sigma_{zy}$. Thus, the Cauchy stress therefore will have six independent terms. Since the Cauchy stress tensor is symmetric, it is fully defined by these six independent stress components instead of the nine terms specified in Eq. 9.47.

The Cauchy stress tensor, $\boldsymbol{\sigma}$, is described as an Eulerian stress formulation since its derivation is with respect to the current material configuration. It is very suitable for small deformation problems such as composite mechanics, viscoelasticity, etc. For problems where finite deformations dominate the material response, we recommend using other alternative measures of stress.

9.5.3 First Piola-Kirchoff Stress Tensor

Also, we can obtain the relationship between the Cauchy stress, $\boldsymbol{\sigma}$, and the first Piola-Kirchoff stress, $\boldsymbol{P}$. To do this, we have to map the area, $\mathrm{d}S$ defined in the reference configuration to the area, $\mathrm{d}s$ defined in the current configuration. The *Nanson's formula* is a useful relationship that accomplishes the mapping from one configuration to another. For the current area, *ds*, the Nanson's formula states:

$$\mathrm{d}s \quad = \quad J\boldsymbol{F}^{-T}\mathrm{d}\boldsymbol{S} \quad \text{where} \quad J = \det\left[\boldsymbol{F}(\boldsymbol{X}, t)\right] \tag{9.48}$$

where $\boldsymbol{F}$ is the deformation gradient tensor and J is the *volume ratio*. If $J = 1$, this suggests that $\det\boldsymbol{F} = 1$ hence there is no motion of the material body. Such a deformation is defined as an *isochoric* or *volume-preserving* deformation.

Recall Eq. 9.45 which establishes that the infinitesimal internal force over an infinitesimal surface element can be defined equivalently in both reference and current configurations. Combining Eqs. 9.45 and 9.46 gives:

$$\boldsymbol{t}\left(\boldsymbol{x}, \boldsymbol{n}, t\right) \mathrm{d}s = \boldsymbol{T}\left(\boldsymbol{X}, \boldsymbol{N}, t\right) \mathrm{d}S \qquad \longrightarrow \qquad \boldsymbol{\sigma}\left(\boldsymbol{x}, t\right) \boldsymbol{n}\mathrm{d}s = \boldsymbol{P}\left(\boldsymbol{X}, t\right) \boldsymbol{N}\mathrm{d}\boldsymbol{S} \tag{9.49}$$

Introducing the Nanson's formula of Eq. 9.48 into Eq. 9.49 yields:

$$\boldsymbol{\sigma}\left(\boldsymbol{x}, t\right) \boldsymbol{n}\left[J\boldsymbol{F}^{-T}\mathrm{d}\boldsymbol{S}\right] \quad = \quad \boldsymbol{P}\left(\boldsymbol{X}, t\right) \boldsymbol{N}\mathrm{d}\boldsymbol{S} \tag{9.50}$$

such that we obtain the expression of the first Piola-Kirchoff stress tensor as a function of the Cauchy stress:

$$\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T} \qquad \text{or} \qquad \boldsymbol{\sigma} = J^{-1}\boldsymbol{P}\boldsymbol{F}^{T} \tag{9.51}$$

The above is described as the *Piola transformation* with the stress in the current configuration transformed to its equivalent representation in the original/reference configuration. Since $\boldsymbol{\sigma}$ is a symmetric matrix, we note that: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, in which case we see that: $\boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{P}^T$.

9.5.4 Kirchoff Stress Tensor

Apart from the Cauchy and first Piola-Kirchoff stress tensors, there exist other alternative measures of stress which are also used in continuum mechanics and the choice of a user depending on the type of problem under investigation. There are advantages and disadvantages associated with each measure of stress a user chooses to use. The *Kirchoff stress tensor*, $\boldsymbol{\tau}$ is one of such alternative stress tensor measures. It represents a multiple of the Cauchy stress and the volume ratio thus:

$$\boldsymbol{\tau} = J\boldsymbol{\sigma}. \tag{9.52}$$

9.5.5 Second Piola-Kirchoff Stress Tensor

A variant of the first Piola-Kirchoff stress tensor is introduced here, the *second Piola-Kirchoff stress tensor*, $\boldsymbol{S}$. It is obtained via the Kirchoff stress tensor according to the following equation:

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{\tau}\boldsymbol{F}^{-T} \quad \text{or also,} \quad \boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T} = \boldsymbol{F}^{-1}\boldsymbol{P} \tag{9.53}$$

where $\boldsymbol{P}$ is the first Piola-Kirchoff stress tensor, $\boldsymbol{\tau}$ is the Kirchoff stress tensor and $\boldsymbol{\sigma}$ is the Cauchy stress tensor. In continuum mechanics, it is widely accepted that $\boldsymbol{S}$ is the *pull-back* of $\boldsymbol{\tau}$ by $\boldsymbol{F}$ and $\boldsymbol{\tau}$ is the *push-forward* of $\boldsymbol{S}$.

The implication of Eq. 9.53 is that the second Piola-Kirchoff stress is symmetric i.e. $\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P} = \boldsymbol{S}^T$. A key relationship is therefore deduced between the first and second Piola-Kirchoff stress tensors:

$$\boldsymbol{P} = \boldsymbol{F}\boldsymbol{S} \tag{9.54}$$

which implies that the mapping function between the two stress measures is defined by the deformation gradient tensor.

9.5.6 Biot Stress Tensor

We define yet another material stress tensor called the *Biot stress tensor*, $\boldsymbol{T}_B$, which is defined in terms of the rotation tensor, $\boldsymbol{R}$, and the second Piola-Kirchoff stress tensor, $\boldsymbol{P}$, thus:

$$\boldsymbol{T}_B = \boldsymbol{R}^T\boldsymbol{P} \tag{9.55}$$

This is a *non-symmetric tensor* and not general ***positive-definite***. We can also re-derive $\boldsymbol{T}_B$ in terms of the right symmetric stretch tensor, $\boldsymbol{U}$. Recall that: $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$ and $\boldsymbol{P} = \boldsymbol{F}\boldsymbol{S}$ as derived previously. If we substitute these into Eq. 9.55, we obtain:

$$\boldsymbol{T}_B = \boldsymbol{R}^T\boldsymbol{P} \quad \longrightarrow \quad \boldsymbol{T}_B = \boldsymbol{R}^T\boldsymbol{F}\boldsymbol{S} = \boldsymbol{R}^T\boldsymbol{U}\boldsymbol{R}\boldsymbol{S} = \boldsymbol{U}\boldsymbol{S} \tag{9.56}$$

Also $\boldsymbol{R}^T\boldsymbol{P} \equiv \boldsymbol{R}\boldsymbol{P}^T$, so another expression of the Biot stress tensor becomes:

$$\boldsymbol{T}_B = \frac{1}{2}\left(\boldsymbol{R}^T\boldsymbol{P} + \boldsymbol{R}\boldsymbol{P}^T\right) \quad \text{or} \quad \boldsymbol{T}_B = \frac{\boldsymbol{R}^T\left(\boldsymbol{V}^{-1}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{V}^{-1}\right)\boldsymbol{R}}{2} \tag{9.57}$$

If we multiply both parts of Eq. 9.55 by $\boldsymbol{R}$, we realize that: $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{T}_B$, which implies that the second Piola-Kirchoff stress tensor, $\boldsymbol{P}$, is related to the Biot Stress tensor, $\boldsymbol{T}_B$, through a mapping function defined by the rotation tensor, $\boldsymbol{R}$. This stress measure is particularly suited for deformations in which finite rotations result during the deformation of the material body like hyperelasticity problems.

9.5.7 Corotated Cauchy Stress Tensor

We illustrated in Fig. 9.6 that for a material body subjected to a deformation gradient tensor, $\boldsymbol{F}$, its deformation comprises stretch tensors i.e. $\boldsymbol{U}$ or $\boldsymbol{V}$, and a rotation tensor, $\boldsymbol{R}$. We showed the polar decomposition of $\boldsymbol{F}$ to be: $\boldsymbol{F} = \boldsymbol{RU} = \boldsymbol{VR}$. In order to achieve the polar decomposition of $\boldsymbol{F} = \boldsymbol{RU}$, the material body was first *rotated* using $\boldsymbol{R}$ before it was stretched using $\boldsymbol{U}$. The result of the first rotation was a rotated material domain, which we called the *intermediate domain*. This intermediate domain is at the heart of a new stress tensor to be defined here.

The *Corotated Cauchy stress tensor*, $\boldsymbol{\sigma}_u$, is determined based on an intermediate domain that is neither the reference nor the current configuration. It is necessary for material responses in which a history of the immediate prior configuration is important.

To derive its values, let us consider Eq. 9.53 and make the Cauchy stress the subject of the formula:

$$\boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T} = \boldsymbol{F}^{-1}\boldsymbol{P} \quad \longrightarrow \quad \boldsymbol{\sigma} = J^{-1}\boldsymbol{FSF}^{T} \tag{9.58}$$

Equation 9.58 implies that the second Piola-Kirchoff stress, $\boldsymbol{S}$, has been operated by a deformation gradient, $\boldsymbol{F}$. If instead of using $\boldsymbol{F}$ to operate at this intermediate domain, we go ahead and operate on it using a stretch tensor according to Fig. 9.6, we end up determining the Corotated Cauchy stress tensor thus:

$$\boldsymbol{\sigma}_u = J^{-1}\boldsymbol{USU}^{T} \tag{9.59}$$

Recall also Eq. 9.53, which we now substitute into Eq. 9.60 such that we obtain:

$$\boldsymbol{\sigma}_u = J^{-1}\boldsymbol{U}\left[J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}\right]\boldsymbol{U}^{T} \quad \longrightarrow \quad \boldsymbol{\sigma}_u = \boldsymbol{R}^{T}\boldsymbol{\sigma}\boldsymbol{R} \tag{9.60}$$

The above derivation of $\boldsymbol{\sigma}_u$ indicates that the original Cauchy stress has been rotated jointed (*corotated*) using the rotation tensor, $\boldsymbol{R}$. Therefore, this makes the formulation of the Cauchy stress now amenable to large deformation problems where finite deformations exists.

9.5.8 Mandel Stress Tensor

Another stress tensor especially for finite plasticity problems is called the *Mandel stress tensor*, Ξ. It is defined in terms of the right Cauchy-Green strain tensor, $\boldsymbol{C}$, and the second Piola-Kirchoff stress tensor, $\boldsymbol{S}$ according to:

$$\Xi = \frac{1}{2}(\boldsymbol{CS} + \boldsymbol{SC}) = \boldsymbol{CS} \quad \longrightarrow \quad \Xi = \boldsymbol{F}^{T}\boldsymbol{FS} \tag{9.61}$$