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ONE-DIMENSIONAL THEORY OF CRACKED BERNOULLI–EULER BEAMS

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Summary—The differential equation and associated boundary conditions for a nominally uniform Bernoulli–Euler beam containing one or more pairs of symmetric cracks are derived. The reduction to one spatial dimension is achieved using integrations over the cross-section after plausible stress, strain, displacement and momentum fields are chosen. In particular the perturbation in the stresses induced by the crack is incorporated through a local function which assumes an exponential decay with distance from the crack and which includes a parameter which can be evaluated by experimental tests. Some experiments on beams containing cuts to simulate cracks are briefly described and the change in the first natural frequency with crack depth is matched closely by the theoretical predictions.

NOTATION

a	depth of crack
A	beam cross sectional area
A_r	reduced beam cross sectional area at crack location
b	half breadth of rectangular beam
$C = m - 1$	
d	half depth of rectangular beam
e	$= (\gamma_{xx} + \gamma_{yy} + \gamma_{zz})$, volume dilatation
E	Young's Modulus of Elasticity
$f(x, z)$	crack function
F_i	body forces
$g_i, \bar{g}_i$	surface tractions and prescribed surface tractions
G	Shear Modulus of Elasticity
$h = d - a$	
$H(h - \lvert z \rvert)$	unit step function at $\lvert z \rvert = h$
I	second moment of area of beam section
I_r	second moment of area of reduced section at crack
$K = \int_A zf \, dA$	
l	length of beam
$L = \int_A f^2 \, dA$	
m	stress magnification factor
p_i	momentum components
p_c	fundamental frequency of crack rectangular beam
p_{BE}	fundamental Bernoulli–Euler frequency of uncracked beam
$P(x, t)$	velocity function
$Q(x) = (I - K)/(I - 2K + L)$	integrated crack function
$r = (\zeta / l)$	non-dimensional longitudinal coordinate
S, S_p, S_u	surfaces of solid
$S(x, t)$	strain function
$T(x, t)$	stress function
$T(p_i)$	kinetic energy density function
$u_i = u, v, w$	displacement components
$\bar{u}_i$	prescribed surface displacements
V	volume of solid
$W(x)$	assumed beam shape function
$W(\gamma_{ij})$	strain energy density function
$x_i = x, y, z$	Cartesian coordinates
X_1, X_2, X_3	integrals over the beam length
α	exponential decay constant
γ_{ij}	strain components
δ_{ij}	Kronecker's delta
ζ	$= x - (l/2)$
κ	shape coefficient in equation (33)
λ	Lamé's Constant
ν	Poisson's ratio
ν_j	direction cosines
ρ	density
τ_{ij}	stress components

1. INTRODUCTION

The dynamic behaviour of structures containing cracks is a subject of considerable current interest in the light of potential developments in the automatic monitoring of structural integrity. It seems plausible that methods might be developed whereby changes in the overall dynamics of a structure, as reflected perhaps in the natural frequencies and associated modal shapes, might be used to indicate the existence of cracks and even give their precise position and extent. Such an approach is not particularly easy because of the relative insensitivity in general of natural frequencies to the presence of a crack unless it is very extensive. Nevertheless such concepts must be examined. Some original work along these lines can be found for example in Ref. [1], more recent related research is described in Refs. [2–4].

As regards the analysis of the problem in the case of a particular structure, numerical procedures, using for instance a finite element modelling, might well be resorted to but it does seem that an understanding of the foundations of the subject could be approached (as it is for uncracked structures) by a fairly fundamental analysis of simple continuous structures such as beams. The work reported in this paper is an attempt to do that for very idealised conditions in a beam in transverse vibration.

Elastic beams are three-dimensional continua but for certain classes of their possible motions they can be modelled at a simpler level. The most elementary of these models, for transverse motion, is the well-known Bernoulli–Euler beam which is the simplest one-dimensional theory currently available. The adjective refers to the fact that in the theory there is only one spatial dimension, measured along the axis of the beam.

In what follows a one-dimensional theory at the same level of approximation as Bernoulli–Euler theory and indeed sharing many of its assumptions, is developed for a straight beam which has at one or more stations along its length open cracks of equal depth originating from its upper and lower surfaces as is more completely described below. The pairs of cracks are taken to be normal to the beam's axis and to be symmetrical about the plane of bending. The reason for this restriction on crack geometry is to hold the particular symmetry associated with bending motion. This symmetry requirement can be relaxed in the present theory with little error as is pointed out later. Also, much more complex theories can be generated which bring out for instance the coupling between flexural and torsional types of motion which would result from a less regular crack but it is intended to present these features in later publications. The reason for specifying an "open" crack is to avoid at this stage the very interesting complexities which result from the nonlinear charactristics presented by a crack which can open and close.

The Bernoulli–Euler beam theory is well known but little understood. The assumptions that are built into it regarding displacement, strain and stress fields are apparently simple but they are at the same time mutually inconsistent. The theory can be derived in a consistent manner only through spatial integrations suggested by the variational theorem of Hu[5], and (independently) Washizu[6] further modified by certain momentum allowances (Barr[7]) which in the present case take into account the neglect of axial (rotatory) inertia. This procedure is a powerful one for developing further complexities or levels of approximation and it is applied here. Essentially, independent plausible assumptions are made for the displacement, strain, stress and momentum fields and these are then fed into the variational statement to produce, after integration over the cross-section of the beam, the equation of motion and its associated boundary conditions.

This equation is not solved directly but instead as an example, the Rayleigh–Ritz method is used to obtain from it an estimate of the drop in the fundamental frequency of a simply supported beam in the presence of a mid-span crack. The theoretical predictions then obtained agree well with results from experiment.

2. VARIATIONAL THEOREM

If displacement components are written as u_i strain components as γ_{ij} and stress components as τ_{ij} with $i, j = 1$, 2, 3 referring to cartesian axes x, y, z and if further p_i are "momentum" components such that $T = \frac{1}{2}\delta_{ij}p_i p_j$ is the kinetic energy density (δ_{ij} is Kronecker's delta) then the extended Hu–Washizu variational principle (Barr[7]) states that for arbitrary independent variations δu_i, $\delta\gamma_{ij}$, $\delta\tau_{ij}$ and δp_i,

$$\int_v \{[\tau_{ij,j}+F_i-\rho\dot{p}_i]\delta u_i+[\tau_{ij}-W_{,\gamma_{ij}}]\delta\gamma_{ij}+[\gamma_{ij}-(1-\tfrac{1}{2}\delta_{ij})(u_{i,j}$$
$$+u_{j,i})]\delta\tau_{ij}+[\rho\dot{u}_i-T_{,p_i}]\delta p_i\}\,\mathrm{d}V+\int_{S_p}[\bar{g}_i-g_i]\delta u_i\,\mathrm{d}S+\int_{S_u}[u_i-\bar{u}_i]\delta g_i\,\mathrm{d}S=0. \quad (1)$$

In this equation $W(\gamma_{ij})$ is the strain energy density function, ρ is the density, F_i and g_i are, respectively, the body forces and the surface tractions, V is the total volume of the solid and S its external surface. The overbarred quantities $\bar{g}_i$ and $\bar{u}_i$ denote the prescribed values of the surface tractions and the surface displacements, respectively, the former act over the surface S_p and the latter over S_u, S_u and S_p together make up the total surface. Differentiation with respect to time is shown by $(\dot{})$. Commas in the subscript denote differentiation in the usual way.

To derive, using equation (1), the Bernoulli–Euler theory for an uncracked beam, it is perhaps simpler to revert to normal engineering notation with $u_1 = u$, $u_2 = v$, and $u_3 = w$ where the x axis is taken along the straight centre line of the beam and the xz plane is the plane of bending. In this notation the displacement field is taken as $u = -zw'$, $v = 0$, $w = w(x, t)$ implying that u is a kinematic consequence of the centre line slope w' (differentiation with respect to x is indicated by $'$). The strain field is taken in the form $\gamma_{xx} = -zS(x, t)$, $\gamma_{yy} = \gamma_{zz} = -\nu\gamma_{xx}$, $\gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0$ where ν is Poisson's ratio. The assumptions for γ_{yy} and γ_{zz} allow anticlastic curvature to develop freely. The stress field is taken to be such that the direct stress along the beam axis is of the form $\tau_{xx} = -zT(x, t)$ while the only other non-zero stress is τ_{xz} which permits the loading of the beam. Finally the momentum or velocity field is assumed to have the form $p_x = p_y = 0$, and $p_z = P(x, t)$. This indicates that longitudinal or rotatory inertia, as it is usually referred to, is neglected as is transverse inertia associated with anticlastic deformation.

In the above S, T, P and w are unknown functions. From equation (1) integrations performed over the beam cross-section assuming independent variations of the unknown functions give the relations $S = w''$, $T = ES$ and $P = \dot{w}$ and, neglecting body forces,

$$EIw^{iv} + A\rho\ddot{w} = 0 \quad (2)$$

where E is Young's Modulus, A is the section area and I its appropriate second moment of area.

In addition the principle generates a full set of boundary conditions but these are omitted here.

3. CRACKED BEAM THEORY

The introduction of cracks will lead to changes in the stress and strain distributions in the vicinity of the cracked section. It is known that near the crack tip there are large stress concentrations and that over the cracked section of the beam the stress is not linearly distributed and all components of stress are likely to be non-zero. However, since the overall dynamics of the beam is of interest it is assumed that the fine structure of the stress distribution is not particularly significant. The equation (1) deals with integrated effects over the section. Of course, if crack propagation modelling was under discussion then this might not be the case and a more refined theory could be required.

The change in stress and strain distribution near the crack are brought in by using a function $f(x, z)$, at present unknown, which has its maximum value at the tip of the crack and which decays with distance from the cracked section. It is applied to the direct stress τ_{xx} only, the remaining direct stresses and the shear stresses out of the plane of bending are still taken to be zero.

It is further assumed that the presence of the crack does not alter in any way the displacement and velocity fields.

The assumptions made following the notation indicated in Section 2 are then, for a nominally uniform beam in the absence of body forces,

$$\begin{aligned}
&u = -zw' \quad v = 0 \quad w = w(x, t)\\
&p_x = 0 \quad p_y = 0 \quad p_z = P(x, t)\\
&\gamma_{xx} = [-z + f(x, z)]S(x, t)\\
&\gamma_{yy} = \gamma_{zz} = -\nu\gamma_{xx}\\
&\gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0\\
&\tau_{xx} = [-z + f(x, z)]T(x, t),\ \tau_{xz} = \tau_{xz}(x, z, t)\\
&\tau_{yy} = \tau_{zz} = \tau_{xy} = \tau_{yz} = 0\\
&F_x = F_y = F_z = 0.
\end{aligned} \quad (3)$$

The inclusion of τ_{xz} is required to permit the loading of the beam as is shown below. The details of its distribution through the depth or along the length of the beam are not required.

These assumptions can now be substituted in the general equation (1) and independent variations of the unknown w, P, S and T considered. For simplicity at this stage it is preferred to consider the variations one by one.

Strain-displacement term

The strain-displacement term of equation (1) becomes for an arbitrary and independent δT variation

$$\int_V \left[\gamma_{xx} - \frac{\partial u}{\partial x} \right] \delta\tau_{xx}\, \mathrm{d}V = \int_x \left\{ \int_A [(-z + f)S + zw''](-z + f)\delta T \, \mathrm{d}A \right\} \mathrm{d}x. \tag{4}$$

This expression (4), defining the various integrals over the section A as

$$I = \int_A z^2 \, \mathrm{d}A, \; K = \int_A zf \, \mathrm{d}A, \; L = \int_A f^2 \, \mathrm{d}A \tag{5}$$

becomes

$$\int_x \{(I - 2K + L)S - (I - K)w''\}\delta T \, \mathrm{d}x. \tag{6}$$

Strain-stress term

The stress-strain term of equation (1) is given by

$$\int_V \left\{ \left[\tau_{xx} - \frac{\partial W}{\partial \gamma_{xx}} \right] \delta\gamma_{xx} - \frac{\partial W}{\partial \gamma_{yy}} \delta\gamma_{yy} - \frac{\partial W}{\partial \gamma_{zz}} \delta\gamma_{zz} \right\} \delta S \; \mathrm{d}V.$$

Substituting the various quantities from equation (1) and using the expression $W = \frac{1}{2}\lambda e^2 + G(\gamma_{xx}^2 + \gamma_{yy}^2 + \gamma_{zz}^2) + \frac{1}{2}G(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2)$ where e is the dilatation $(\gamma_{xx} + \gamma_{yy} + \gamma_{zz})$ and λ is Lamé's constant, the expression simplifies to the form,

$$\int_x \{(T - ES)(I - 2K + L)\}\delta S \, \mathrm{d}x. \tag{7}$$

Velocity-momentum term

In a similar way the velocity momentum is written using assumptions (3) as

$$\int_x (\rho A\dot{w} - \rho PA)\delta P \, \mathrm{d}x. \tag{8}$$

Dynamic equilibrium term

The first term of equation (1) leads to the equation of motion. Using assumptions (3) the term in question is

$$\int_V \left[\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \right) \delta u + \left(\frac{\partial \tau_{xz}}{\partial x} - \rho \dot{P}_z \right) \delta w \right] \mathrm{d}V = \int_A \int_x \left\{ \left([(-z + f)T]' + \frac{\partial \tau_{xz}}{\partial z} \right) (-z\delta w') + \left[\frac{\partial \tau_{xz}}{\partial x} - \rho \dot{P} \right] \delta w \right\} \mathrm{d}x \, \mathrm{d}A. \tag{9}$$

The first term of (9) can be integrated by parts in the following way. Noting that $\delta w' \equiv \delta(\partial w/\partial x) = \partial(\delta w)/\partial x$, we have

$$\int_A \int_x \left\{ \left(-z[(-z + f)T]' - z\frac{\partial \tau_{xz}}{\partial z} \right) \frac{\partial}{\partial x}(\delta w) \right\} \mathrm{d}x \, \mathrm{d}A = \int_A \left\{ -z[(-z + f)T]' - z\frac{\partial \tau_{xz}}{\partial z} \right\} \delta w \, \mathrm{d}A \Big|_x$$
$$- \int_A \int_x \left\{ -z[(-z + f)T]'' - \frac{\partial}{\partial x}\left(z\frac{\partial \tau_{xz}}{\partial z} \right) \right\} (\delta w) \, \mathrm{d}x \, \mathrm{d}A. \tag{10}$$

The last term of expression (10) can be integrated by parts over z as follows,

$$\int_x \frac{\partial}{\partial x} \left\{ \int_y \int_z z\frac{\partial \tau_{xz}}{\partial z} \mathrm{d}z \, \mathrm{d}y \right\} \delta w \, \mathrm{d}x = \int_y \int_x \frac{\partial}{\partial x}(z\tau_{xz})\delta w \, \mathrm{d}x \, \mathrm{d}y \big|_z - \int_A \int_x \frac{\partial \tau_{xz}}{\partial x} \delta w \, \mathrm{d}x \, \mathrm{d}A.$$

Expression (10) thus contains the following boundary terms, which are incorporated with the other boundary conditions, of equation (1)

$$\int_A \left\{ -z[(-z + f)T]' - z\frac{\partial \tau_{xz}}{\partial z} \right\} \delta w \, \mathrm{d}A \big|_x + \int_y \int_x \frac{\partial}{\partial x}(z\tau_{xz})\delta w \, \mathrm{d}x \, \mathrm{d}y \big|_z. \tag{11}$$

The remaining two terms of expression (10) incorporated in (9) reduce that expression to the form

$$\int_x \int_A \{[(-z^2 + zf)T]'' - \rho\dot{P}\} \, \delta w \, \mathrm{d}A \, \mathrm{d}x. \tag{12}$$

Performing the double differentiation indicated and integrating over the cross-section allows this to be rewritten as

$$\int_x [K''T + 2K'T' + (K - I)T'' - \rho A\dot{P}]\delta w \, \mathrm{d}x. \tag{13}$$

Boundary conditions

As regards the surface integrals over S_p and S_u in equation (1), it will be assumed that the lateral surfaces of the beam are free of external traction, i.e. $\bar{g}_i = 0$. The surface force is obtained from the stress components by the expression $g_i = \tau_{ij}\nu_j$ where the ν_j are the direction cosines of the external normal to the surface with the co-ordinate directions. If the beam is untapered the normal to its lateral surfaces will be at right angles to its axis so that ν_x is zero. Thus the expressions for the g_i using assumptions (3) are

$$\left.\begin{aligned} g_x &= \tau_{xx}\nu_x + \tau_{xy}\nu_y + \tau_{xz}\nu_z = \tau_{xz}\nu_z \\ g_y &= \tau_{xy}\nu_x + \tau_{yy}\nu_y + \tau_{yz}\nu_z = 0 \\ g_z &= \tau_{zx}\nu_x + \tau_{zy}\nu_y + \tau_{zz}\nu_z = 0. \end{aligned}\right\} \tag{14}$$

On the other hand, over the ends of the beam $x = 0, l$, $\nu_x = -1$ and $\nu_x = +1$, respectively (assuming plane ends normal to the beam axis) and from the general expressions in (14) g_x reduces to $\pm\tau_{xx}$ and g_z to $\pm\tau_{xz}$. The specified forces $\bar{g}_i$ at the ends, integrated over the section, correspond to an applied force or moment.

The surface integrals in equation (1) thus take the form, over the surface of the beam at the limits of z (say z_1 and z_2, $z_2 > z_1$)

$$\int_x\int_y \{[0 - \tau_{xz}]_{z=z_2}\,\delta u + [0 + \tau_{xz}]_{z=z_1}\,\delta u\}\,dy\,dx.$$

This can be written

$$\left[\int_x\int_y -\tau_{xz}\,\delta u\,dy\,dx\right]_{z_1}^{z_2}$$

then using the relation $\delta u = -z\delta w'$ and integrating by parts over x this becomes

$$\left[\left(\int_y z\tau_{xz}\,\delta w\,dy\right)\bigg|_x - \int_y\int_x \frac{\partial}{\partial x}(z\tau_{xz})\delta w\,dx\,dy\right]_{z_1}^{z_2}. \tag{15}$$

The second term of this expression cancels with the final term of expression (11). The second term of (11) can be integrated by parts over z and results in a term which cancels the first of expression (15). The remaining terms of (11) apply to the boundaries of x and are

$$\int_A \{-z[(-z + f)T]' + \tau_{xz}\}\,\delta w\,dA\big|_x. \tag{16}$$

Similarly, the surface integrals of equation (1) over the ends of the beam ($x = 0$ and $x = l$) take the form, when forces $\bar{X}$ and $\bar{Z}$ are prescribed:

$$\left[\int_A \{(\bar{X} - \tau_{xx})\delta u + (\bar{Z} - \tau_{xz})\delta w\}\,dA\right]_{x=l} + \left[\int_A \{(\bar{X} + \tau_{xx})\delta u + (\bar{Z} + \tau_{xz})\delta w\}\,dA\right]_{x=0}. \tag{17}$$

Incorporating expression (16) with expression (17) and substituting for the quantities δu, δw and τ_{xx} from the assumed functions in equation (3), integration over the section can be performed and the resulting boundary terms take the form,

$$\left[\left\{-\int_A z\bar{X}\,dA + (K - I)T\right\}\delta w' + \left\{\int_A \bar{Z}\,dA + (I - K)T' - K'T\right\}\delta w\right]_{x=l}$$
$$\left[\left\{-\int_A z\bar{X}\,dA - (K - I)T\right\}\delta w' + \left\{\int_A \bar{Z}\,dA + (I - K)T' - K'T\right\}\delta w\right]_{x=0}. \tag{18}$$

On the other hand, when displacements $\bar{u}$ and $\bar{w}$ are prescribed, the surface integrals of equation (1) over the ends $x = 0, l$ are

$$\left[\int_A \{(u - \bar{u})\delta\tau_{xx} + (w - \bar{w})\delta\tau_{xz}\}\,dA\right]_{x=l} - \left[\int_A \{(u - \bar{u})\delta\tau_{xx} + (w - \bar{w})\delta\tau_{xz}\}\,dA\right]_{x=0}.$$

After substitution for u, w and τ_{xx} from equation (3) and integration over the section, this expression becomes,

$$\left[\left\{(I - K)w' - \int_A \bar{u}(f - z)\,dA\right\}\delta T + \{(w - \bar{w})A\}\delta\tau_{xz_{x=l}}\right.$$
$$-\left[\left\{(I - K)w' - \int_A \bar{u}(f - z)\,dA\right\}\delta T + \{(w - \bar{w})A\}\delta\tau_{xz}\right]_{x=l} \tag{19}$$

Derived equations of the problem

The entire variational statement for the problem can now be assembled using the equation (1) and the various terms of the variational expressions (6)–(8) and (13) along with the boundary terms given in expressions (18) and (19).

The variations δw, $\delta w'$, δP, δS, δT and $\delta\tau_{xz}$ are regarded as independent so that equation (1) implies, for arbitrary values of these variations, that each expression multipled by them in the volume inegral must independently be zero. This gives the following relations directly.

From expression (6) for δT,

$$S = Q(x)w'' \tag{20}$$

where

$$Q(x) = (I - K)/(I - 2K + L). \tag{21}$$

From expression (7) for δS,

$$T = ES. \tag{22}$$

From expression (8) for δP,

$$P = \dot{w} \tag{23}$$

From expression (13) for δw,

$$K''T + 2K'T' + (K - I)T'' - \rho A\dot{P} = 0. \tag{24}$$

Equation (24) is the equation of motion. Substitution for T and P can be made in terms of the displacement w using equations (20)–(23). The resulting equation is

$$E(I - K)Qw^{iv} + 2E[Q'(I - K) - K'Q]w''' + E[Q''(I - K) - 2K'Q' - K''Q]w'' + \rho A\ddot{w} = 0. \tag{25}$$

The boundary conditions appropriate to the equation of motion (25) are obtained by equating the surface integral expression (18) to zero in the case of prescribed external forces and the equivalent expression (19) to zero in the case of prescribed displacements.

Thus, for example a cantilever beam with its fixed end at $x = 0$, has at that point displacements $\bar{u}$ and $\bar{w}$ prescribed as zero while at $x = l$ it has $\bar{X}$ and $\bar{Z}$ prescribed as zero.

At $x = 0$ equation (19) then gives directly

$$w = 0, w' = 0.$$

At $x = l$ equation (18) gives $(K - I)T = 0$ and $(I - K)T' - K'T = 0$.

The first of these using equations (20) and (22) implies, from $T = 0$, that $w'' = 0$. The second then implies $T' = 0$ or $w''' = 0$ as expected.

If the crack is absent from the beam then the functions K and L are zero and $Q(x)$ becomes unity. The equation of motion (25) and the boundary conditions become those of the uniform Bernoulli–Euler beam.

The crack function f(x, z)

The above theory is based on stress and strain distributions (3) which are affected by the presence of the crack or cracks through the function $f(x, z)$. It is necessary to establish a plausible means of evaluating f for a given cross-section and crack geometry.

It is assumed in the first place that cracks occur in symmetric pairs one from the upper surface $z = -d$ of the beam and the other from its lower surface $z = d$, both at the same value of x. A typical situation is shown in Fig. 1.

The cracks are of depth a and the remaining material between them is of depth $2h$. Remote from the crack the distribution of the stress τ_{xx} is known to be linear to good approximation for relatively long wavelengths as is assumed in the application of Bernoulli–Euler theory.

At the cracked section a linear distribution of τ_{xx} across the reduced section is again assumed (although this is known not to be correct) but the magnitude of the stress level at a given co-ordinate z is augmented. A simple though approximate way of estimating the magnitude of this increase which should be reasonable at least for

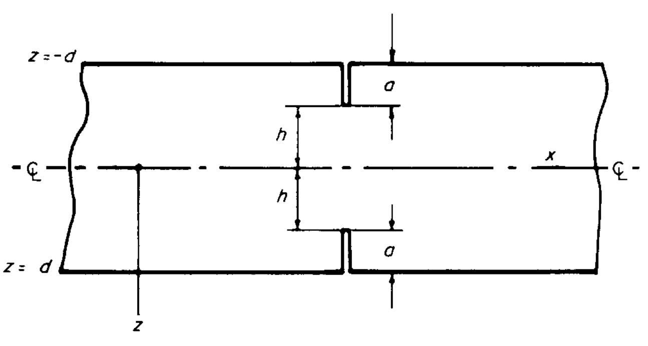

Fig. 1. Typical symmetric crack geometry.

shallow cracks is to demand that the same bending moment be carried by the cracked section as would have been carried by the intact section.

Thus,

$$\int_A (-zT)z \, \mathrm{d}A = \int_{A_r} (-z + f(x_c, z))Tz \, \mathrm{d}A \tag{26}$$

where A_r is the reduced section remaining at the crack ($x = x_c$).

The distribution $(-z + f(x_c, z))$ is, as has been stated, taken to be linear in z say $(-mz)$ with m constant. Hence, equation (26) becomes

$$-T\int_A z^2 \, \mathrm{d}A = -mT\int_{A_r} z^2 \, \mathrm{d}A$$

so that

$$m = (I/I_r) \tag{27}$$

where I_r is the second moment of area of the reduced section. Further at the cracked section the stress τ_{xx} drops stepwise to zero when the cracked zone is entered, i.e. for $|z| > h$. This is introduced in $f(x, z)$ through the use of the unit step function $H(h - |z|)$ at $z = h$.

The above accomodates the z-wise distribution of τ_{xx}. In the direction x, on the other hand, the stress τ_{xx} is assumed to decay from its maximum value at the cracked zone to its nominal (Bernoulli–Euler) value remote from the crack. This question of the decay from the cracked zone is similar to questions in the application of the Principle of St. Venant. In this case however rather than a load distribution being changed over some defined area, we have an unchanged load (zero, for the unloaded beam) but the external surface of the beam is being changed through the formation of the crack surfaces.

In many instances where analytical solutions relating to St. Venant's Principle are known the decay rates are found to be exponential and this is also assumed for the present problem.

Thus, taking the crack to be at the position $x = x_i$, the function $f(x, z)$ occuring in the stress distribution τ_{xx} (equation (3)) is taken in the form

$$f(x, z) = [z - mz\,H(h - |z|)]\exp(-\alpha|x - x_i|/d) \tag{28}$$

where d is half the depth of the beam section.

In equation (28), α is a positive non-dimensional constant which may be determined from experimental results. The use of $|x - x_i|$ gives a symmetric decay on either side of the crack. A sketch of the assumed stress distribution τ_{xx} near the crack is shown in Fig. 2 for a rectangular section beam. Boundary terms from the variational principles (1) which arise at the new crack surfaces can be shown to be mutually cancelling.

Using equation (28), the stress distribution can be written in a general way for the case in which the beam has symmetric cracks at several points x_i ($i = 1, 2, 3, \ldots, n$), thus,

$$\tau_{xx}(x, z, t) = \{-z + [z - mz\,H(h - |z|)]\sum_i^n \exp(-\alpha|x - x_i|/d)\}\,T(x, t) \tag{29}$$

4. RECTANGULAR SECTION BEAMS

As an example of the application of the above theory attention is now focussed on the case of a beam of rectangular section of depth $2d$ and breadth $2b$ with a symmetric pair of cracks at mid-span $x = l/2$. The constants I, I_r, K, L, Q and m of equations (5), (21) and (27) can be evaluated. They are found to be

$$I = 4bd^3/3, \quad I_r = 4bh^3/3, \quad m = (d/h)^3, \quad K = 0$$

and

$$L = CI\exp(-2\alpha|x - (l/2)|/d) \quad \text{where} \quad C = (m - 1)$$

also

$$Q = [1 + C\exp(-2a|x - (l/2)|/d)]^{-1}. \tag{30}$$

The equation of motion of the unloaded beam from equation (25) is

$$EIQ\,w^{\mathrm{Iv}} + 2EIQ'w''' + EIQ''w'' + \rho A\ddot{w} = 0. \tag{31}$$

This differential equation has to be considered in conjunction with appropriate boundary conditions taken from equations (18) and (19) along with any relevant initial conditions.

Approximate determination of first natural frequency

The application of the describing differential equation (31) to a real beam can be examined by considering the particular case of the variation of the first natural frequency of a simply supported beam having a central symmetric crack of varying depth. This problem can be tackled experimentally as described below, and can be approached most easily analytically by using the Rayleigh–Ritz method.

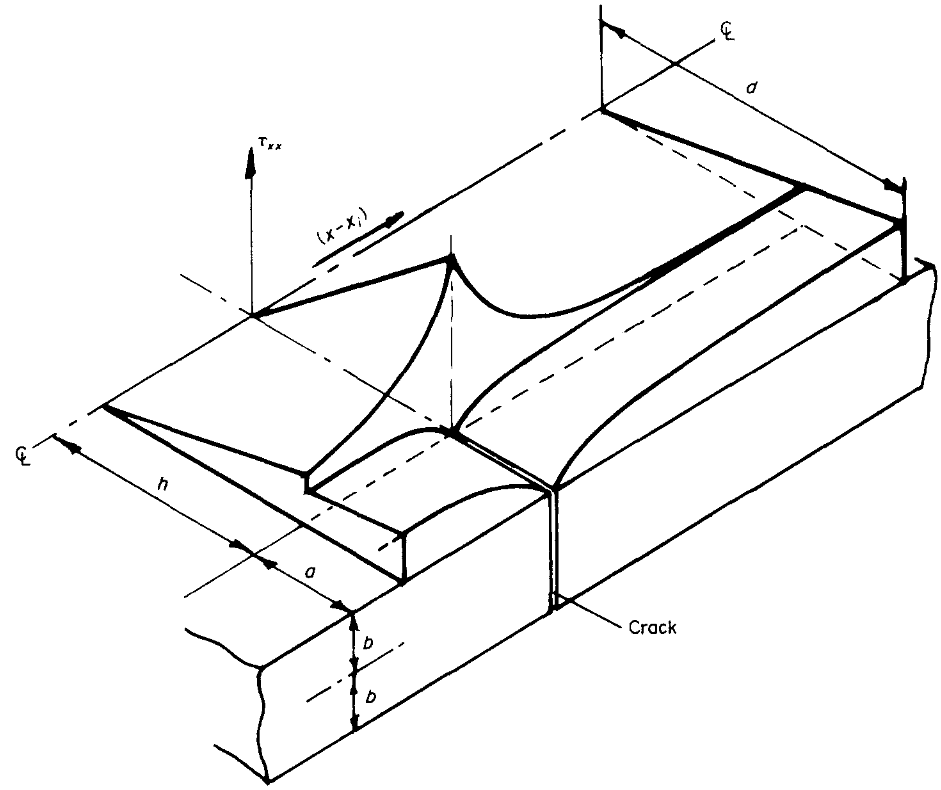

FIG. 2. Sketch of assumed distribution of stress τ_{xx} in vicinity of crack.

The Rayleigh quotient associated with equation (31) can be written in the form

$$(p_c/p_{BE})^2 = (l/\pi)^4 \int_0^l Q(W'')^2\,\mathrm{d}x \Big/ \int_0^l W^2\,\mathrm{d}x \tag{32}$$

where p_c is the natural frequency of the cracked beam, p_{BE} is the Bernoulli–Euler frequency of the uncracked beam and $W(x)$ is an assumed shape function. This equation can be arrived at by equating potential and kinetic energies in an assumed periodic motion of frequency p_c.

For the first mode of a simply supported beam the shape function is taken (for $0 \leqslant x \leqslant l/2$) as

$$W(x) = \sin(\pi x/l) + \kappa\{(x/l) - [4(x/l)^3/3]\}. \tag{33}$$

The first term in this expression is that of the uncracked beam. The second term, multiplied by the unknown factor κ, is intended to account for the presence of the crack at $x = l/2$, it has zero slope at that point. The stated expression applies only up to $x = l/2$ and its mirror image applies from $l/2$ to l. The displacement function is thus symmetric about the beam centre and only half the beam $(0 \leqslant x \leqslant l/2)$ need, therefore, be considered.

Since equation (32) gives an upper bound on p_c the factor κ is evaluated by minimizing p_c. In evaluating expression (32) it is convenient to change co-ordinates to the centre of the beam using $\zeta = x - (l/2)$; Q is given by equation (30).

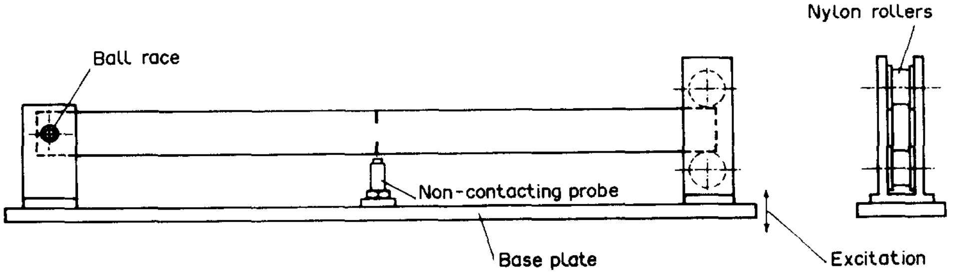

FIG. 3. Experimental beam and supports.

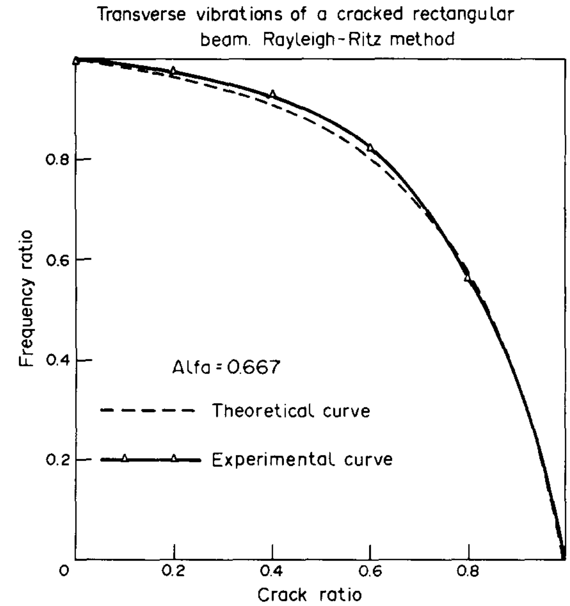

FIG. 4. Comparison of experimental and theoretical values of frequency ratio.

Substituting equation (33) in equation (32) results in

$$(p_c/p_{BE})^2 = [X_1 + (16X_2\kappa/\pi^2) + (64X_3\kappa^2/\pi^4)]/[0.25 + (16\kappa/\pi^4) + 0.02698\kappa^2] \quad (34)$$

where X_1, X_2 and X_3 are the integrals,

$$X_1 = \int_0^{1/2} Q\cos^2\pi r\,\mathrm{d}r,\quad X_2 = \int_0^{1/2} (\tfrac{1}{2} - r)Q\cos\pi r\,\mathrm{d}r,\quad X_3 = \int_0^{1/2} (\tfrac{1}{2} - r)^2 Q\,\mathrm{d}r,$$

where $r = \zeta/l$ and $Q = [1 + C\exp(-2\alpha rl/d)]^{-1}$. These integals are evaluated numerically and the expression for $(P_c/p_{BE})^2$ is minimized with respect to κ.

This evaluation was carried out for a range of crack depths, defined by the parameter C in Q. The results depend on the value assumed for α, a value that gave reasonable comparison with experimental data was chosen.

Experimental work and comparison with theory

An experimental beam of mild steel of length 0.575 m and of rectangular section 31.75 mm depth and 9.525 mm breadth was attached to a base plate through a pin running in a ball race at one end and through a pair of nylon rollers bearing on the upper and lower surfaces of the beam at the other end (see Fig. 3.). This was an attempt to simulate simply supported boundary conditions. The base plate was attached to the vibrating table of a large electrodynamic shaker. A non-contacting probe detected the motion of the beam relative to the table. The table vibration was itself monitored by an attached accelerometer. The power amplifier for the shaker was driven by a precision oscillator with a digital frequency readout.

The natural frequencies in flexure of the uncracked beam were first obtained. Then fine saw cuts, symmetric with respect to the neutral axis and normal to it were made on the upper and lower beam surfaces at midlength. The natural frequencies of the beam for that particular depth of cut were measured and the process was repeated until the cuts went through the whole depth of the beam. This was repeated for a number of beams of the same dimensions.

Experimental and analytical results are compared in Fig. 4. They are presented in the form of the frequency ratio, that is the ratio of the frequency of the cracked beam to that of the uncracked case, against the crack depth ratio, i.e. the ratio of the depth of cut a to the half depth of the beam d. The experimental points are averages from tests but in general the spread of observations about the points was small.

The theoretical curve is based on the expression (34) with the parameter α chosen to give good agreement with the experimental points. In general terms a higher value of α will underestimate the frequency reduction due to the crack and inversely for a lower α.

5. CONCLUSIONS

The one-dimensional theory for the flexural motion of a Bernoulli–Euler beam containing one or more pairs of symmetric cracks which is presented in the foregoing, provides among other things a means for estimating the frequency changes brought about by the existence of the cracks. The theory depends on modelling the stress field perturbation induced by the crack in an approximate way which includes an exponential decay parameter

α which has been estimated from the experimental data on natural frequency change as a function of crack depth.

Simple predictions of this frequency change from the theory have been found to show close agreement with the experimental data and give confidence for the application of the theory to the general dynamics of the beam.

The theory can be extended to situations in which the crack symmetry is absent and in which coupling between the various forms of motion such as bending and torsion takes place. Such coupling is not generally significant except in regions of the spectrum where for instance both the frequencies of predominantly bending and predominantly torsional motion and their corresponding wavelengths are approximately the same. Away from these regions the theory that has been presented above can also be applied to the flexural vibration of a beam with a crack on one side only. It is intended to present the theories for coupled motion of cracked beams at a later date.

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