Flow of Non-Newtonian Fluids—Correlation of the Laminar, Transition, and Turbulent-flow Regions

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All available data on flow of non-Newtonians in pipes have been correlated on the conventional friction factor — Reynolds number plot for Newtonian fluids. This correlation, theoretically rigorous in the laminar flow region, was tested with data on 16 different non-Newtonian materials covering the 2.1 $\times$ 10^9 range of Reynolds numbers from 6.3 $\times$ 10^{-5} to 1.3 $\times$ 10^5. Pipe diameters varied from 1/8 to 12 in. As the correlation does not depend on the type of fluid encountered, it may be used with Newtonian and non-Newtonian fluids alike.

In spite of the great range of the available experimental data, further work is necessary in the transition and turbulent-flow regions. No data at all were available on thixotropic, rheopectic, and dilatant fluids, and extension of the correlation to these materials should prove most illuminative from both theoretical and practical viewpoints.

Workers in the field of rheology have long classified non-Newtonian fluids as plastic (or Bingham plastic), pseudoplastic, dilatant, thixotropic, and rheopectic (*2, 9,* and *16*), and a few others, having found the foregoing divisions unsatisfactory, have added "general non-Newtonian" classifications (*12, 22,* and *23*). Engineering design procedures have then been developed on this basis, particularly for the first of the aforementioned fluid types (*4, 10, 12,* and *18*).

The classification of fluids into those categories constitutes a gross oversimplification of the facts. It has repeatedly been shown (*5, 6, 21,* and *26*) that the classification into which a fluid falls, and even the numerical values assigned its rheological properties, is extremely dependent upon the experimental conditions under which the measurements are made. Under certain narrow ranges of shear rate, for example, a given fluid may clearly appear to behave as a Bingham plastic; at slightly different rates of shear the pseudoplastic relationship is closely followed and, particularily at high shear rates, the same material may appear almost Newtonian.

The important consequences of these facts are two in number: first, large extrapolations of data to new conditions are not permissible where this system of classification is used and, second (and most important), design procedures for prediction of pressure drop in pipe lines become astronomically complex if they must be changed every time the fluid velocity in a pipe line (shear rate) is changed, which may be the case if the fluid exhibits a different type of flow behavior over every new region of shear rates.

It is obvious from the foregoing discussion that some method must eventually be developed which is universally applicable to all fluids —Newtonian and non-Newtonian alike. Several attempts to do this have been reported in the literature (*2, 16, 24,* and *28*). The second of these is limited because of its empirical nature and the last two require the assumption of equations relating fluid shear rate to shear stress. This is an eminently more useful procedure than the arbitrary classification of fluids into rheological types, but nevertheless these equations do not always correlate fluid properties with adequate precision. In addition, the first and last of these prior-art procedures are of considerable complexity. Therefore one major purpose of the present work was to provide a design procedure which might be completely general and rigorous yet as simple in form as the standard friction factor—Reynolds number correlations for Newtonian fluids.

The second severe limitation of these general prior-art methods is their unproved ability to predict accurately the point of onset of turbulence. In a few industries the non-Newtonian fluids encountered are invariably viscous pastes and this restriction is not important; in other industries turbulent flow is of common occurrence, particularily where it is needed to produce reasonably high rates of heat or mass transfer. The second major purpose of the present investigation, therefore, was to provide at least a tentative criterion for the onset of turbulence and to correlate all the available data on flow of non-Newtonian fluids outside the laminar flow region.

Very few of the prior-art publications in engineering deal with thixotropic or rheopectic fluids. It is also necessary to exclude these fluids from consideration in this work, but, as pointed out previously (*16*), this is not a serious limitation at the present time.

DEVELOPMENT OF CORRELATION

Rabinowitsch (*20*) developed an expression for the rate of shear of a fluid which is entirely independent of the fluid properties, provided thixotropy and rheopexy are eliminated. The complete development of this equation was also presented in a paper of Mooney (*17*). Their final expression takes the form

$$\left(\frac{-du}{dr}\right)_w = 3\left(\frac{8Q}{\pi D^3}\right) + \frac{D\Delta P}{4L} \cdot \frac{d\,(8Q/\pi D^3)}{d\,(D\Delta P/4L)} \qquad (1)$$

Since the bulk velocity V is equal to $4Q/\pi D^2$, Equation (1) can be rearranged as

$$\left(\frac{-du}{dr}\right)_w = \frac{3}{4}\left(\frac{8V}{D}\right) + \frac{1}{4}\left(\frac{8V}{D}\right) \frac{d\,(8V/D)\,(8V/D)}{d\,(D\Delta P/4L)/D\Delta P/4L} \qquad (2)$$

$$= \frac{3}{4}\left(\frac{8V}{D}\right) + \frac{1}{4}\left(\frac{8V}{D}\right) \frac{d \ln (8V/D)}{d \ln (D\Delta P/4L)} \qquad (3)$$

In order to simplify Equation (3), the derivative will be denoted by the symbol $1/n'$. Rearrangement of Equation (3) then gives

$$\left(\frac{-du}{dr}\right)_w = \frac{3n' + 1}{4n'} \frac{8V}{D} \qquad (4)$$

It must be emphasized that Equation (4) is simply another form of Equations (1) to (3) and is therefore an entirely general expression of the relationship between rate of shear $(-du/dr)$ and bulk flow rate of the fluid. It is preferable to the original Rabinowitsch relationship [Equation (1)] for two reasons, however:

1. It is in a simpler, more compact form.
2. The derivative n' represents the slope of a logarithmic plot of $D\Delta P/4L$ vs. $8V/D$ and has been found to be very nearly a constant over wide ranges of shear stress for a great variety of non-Newtonian fluids. From a calculational viewpoint it is much easier, therefore, to work with this parameter than with the derivative in Equation (1). An equation similar to Equation (4) was also developed by Schofield (*25*).

The definition of n' will next be rearranged to show the relationship of this physical property to other better known fluid properties.

Since

$$\frac{1}{n'} = \frac{d \ln 8V/D}{d \ln D\Delta P/4L}$$

one may write (over any range of shear stresses for which n' is constant)

$$\frac{D\Delta P}{4L} = K'\left(\frac{8V}{D}\right)^{n} \qquad (5)$$

where K' is also a constant. It has been found experimentally that for most fluids K' and n' are constant over wide ranges of $8V/D$ or $D\Delta P/4L$. For some fluids this is not the case, however, and care must be taken to ensure that the range of integration is small, i.e., that the particular values of K' and n' used are valid for the actual $8V/D$ or $D\Delta P/4L$ with which one is dealing in a given design problem. Conceivably in the limiting case different values of K' and n' would have to be used for every value of $8V/D$ [in this case Equation (5) would be the equation of the tangent to the curve at a single point] but it must be emphasized that for almost all fluids the reverse is true and K' and n' are constant over wide ranges of $8V/D$.

On substituting for $8V/D$ in Equation (5) from Equation (4) and denoting the shear stress at the wall of a pipe $(D\Delta P/4L)$ by T_w, one obtains

$$T_w = K'\left(\frac{4n'}{3n'+1}\right)^{n'}\left(\frac{-du}{dr}\right)_w^{n'} \qquad (6)$$

We are now in a position to appreciate the significance of the physical property n'. If it is a constant with the value of unity, Equation (6) becomes

$$T_w = K'\left(\frac{-du}{dr}\right)_w \qquad (7)$$

That is to say, the familiar linear relationship between shear stress and shear rate of Newtonian fluids appears, and K' is obviously equal to μ/g_c. If, on the other hand, n' is less than unity (but still constant) one obtains the Ostwald equation for pseudoplastic fluids, viz.,

$$T_w = K\left(\frac{-du}{dr}\right)^{n} \qquad (8)$$

Similarily, if n' is greater than unity the fluid is dilatant in character, a class of which the common starches are outstanding examples. (References *2*, *9*, and *16*, among others, discuss these various types of non-Newtonian fluids in some detail.)

In summary of the preceding paragraphs it is seen that *the coefficient* n' *is that physical property of a fluid which characterizes its degree of non-Newtonian behavior*: the greater the divergence of n' from unity (in either direction), the more non-Newtonian is the fluid in question. It is believed that this may be the first time that a quantitative and rigorous scale has been proposed by means of which the degree of non-Newtonian behavior of all fluids (other than time-dependent ones) may be established and compared.

Equation (5) is the basic relationship for relating pressure drop to flow rate by means of geometric parameters and the two physical properties of the fluid, K' and n'. Whereas n' defines the degree of non-Newtonian behavior of the fluid, K' defines its consistency: the larger the value of K' the "thicker" or "more viscous" the fluid.

The next step in the mathematical development is to relate ΔP in Equation (5) to the Fanning friction factor, in order to enable the computation of this parameter, and therefore for all fluids the Reynolds numbers and flow rates at which stable laminar flow no longer is found. The usual definition (*30*) of the former may be written as

$$f = \frac{D\Delta P}{4L} \Big/ \frac{\rho V^2}{2g_c} \qquad (9)$$

Substitution of $D\Delta P/4L$ from Equation (5) into Equation (9) gives

$$f = \frac{16\gamma}{D^{n'}\, V^{2-n'}\rho} \qquad (10)$$

where

$$\gamma = g_c K' \, 8^{n'-1} \qquad (11)$$

By letting $f = 16/N_{Re}$ as for Newtonian fluids in laminar flow, one defines a generalized Reynolds number:

$$N_{Re} = \frac{D^{n'}\, V^{2-n'}\rho}{\gamma} \qquad (12)$$

The significance of the foregoing equations cannot be overemphasized. They state that all fluids must follow the usual f vs. N_{Re} relationship in the laminar-flow region when one uses the generalized Reynolds number defined by Equation (12). As the only implicit assumption in this development is that of no "slip" at the wall of the pipe, this development is completely rigorous and may, in fact, be used to check the accuracy of experimental data. If perfect coincidence with the $f = 16/N_{Re}$ line is not obtained in the laminar-flow region, either the data or calculations are in error or the fluid exhibits evidence of thixotropic or rheopectic behavior.

For Newtonian fluids, $n' = 1.000$, $K' = \mu/g_c$, γ reduces to μ (the viscosity of the fluid), and N_{Re} [Equation (12)] reduces to the familiar $DV\rho/\mu$, showing that this traditional dimensionless group is merely a special restricted form of the more general one proposed here.

In order to make use of these relationships, it is necessary to obtain K' and n' for the fluid being considered. The easiest and the only perfectly rigorous method is to measure the pressure drop and flow rate of the material in any good capillary-tube viscometer and to apply Equation (5) to these data. Since K' and n' were independent of shear rate for all the fluids on which literature data were available, pressure-drop determinations at two flow rates are theoretically sufficient completely to define the physical properties of the fluid. (It may be noted in passing that two experimental measurements are the minimum number which must be made on any non-Newtonian fluid in order to define its rheological properties adequately, as compared with the single measurement which suffices for Newtonians.) From a practical viewpoint, however, it is usually preferable to take more data because of the experimental difficulty of obtaining accurate, reproducible data on many non-Newtonian systems.

Rotational viscometers are also generally satisfactory for evaluation of K and n by means of Equation (8). [Krieger and Maron (*13*) have shown how $(-du/dr)_w$ may be obtained from data taken on such an instrument.) It must be noted, however, that K and n can be related to K' and n' only when these properties are constant over a reasonably wide (say, ten-fold) range of shear stresses. This has not been found to be a significant limitation on work to date but may be with certain other non-Newtonian materials. Accordingly, the over-all recommendation is to take data with a capillary-tube viscometer whenever this is convenient.

TRANSITION FROM LAMINAR TO TURBULENT FLOW

Although deviation from purely laminar or streamline motion has been observed to occur at Reynolds numbers as low as about 1,000 for Newtonian fluids (*19*), the formation of truly persistent eddies and substantial deviation of the velocity profile from the parabola of the laminar region occur rather suddenly over a narrow range of Reynolds numbers near 2,100 (*14* and *19*). Since the Reynolds number has been empirically found to define this transition region for Newtonian fluids, many workers have attempted to justify theoretically its use as a criterion for onset of turbulence by considering the Reynolds number to be a ratio of "inertial forces to viscous forces." It has been pointed out elsewhere (*19*) that this concept seems to be of little theoretical value.

The same is not true of the Fanning friction factor. As shown by Equation (9), it is indeed a ratio of forces—the viscous shear force (per unit wall area) divided by the average main-stream inertial force (per unit cross-sectional area). In the present work, therefore, f rather than N_{Re} will be generally used as the criterion for onset of turbulence. The two apparent disadvantages of such an approach are not believed to be of great importance: first, the problem of an increasing f in the transition region (at least for Newtonian fluids) must be noted only to avoid possible ambiguity, and the second shortcoming, that the inertial forces are average values, is common to almost all approaches of this type (since the various types of non-Newtonians will exhibit different velocity profiles, the averaging procedure is not uniform). In view of the unique relationship between f and N_{Re} for all fluids in the present correlation, it is still impossible to distinguish which of these criteria is rigorously correct.

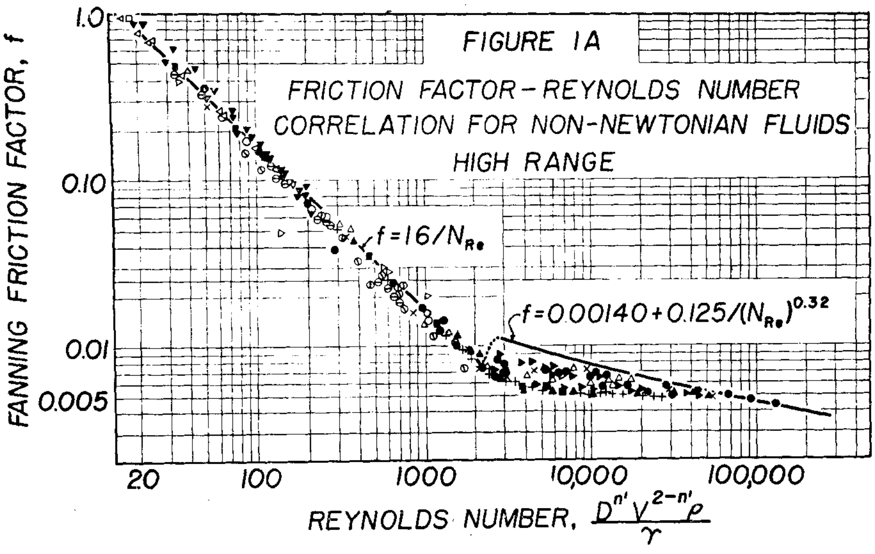

Fig. 1A. Friction-factor–Reynolds number correlation for non-Newtonian fluids—high range.

TABLE 1.—RHEOLOGICAL CONSTANTS FOR FLUIDS SHOWN IN FIGURE 1

Symbol used in Figure 1	Nominal pipe size, in.	Composition of fluid	Rheological constants n'	γ	Source of data
+	1	23.3% Illinois yellow clay in water	0.229	0.863	(*4*)
⦶	⅞ and 1½	0.67% Carboxy-methyl-cellulose (CMC) in water	0.716	0.121	(*24*)
⊖	⅞ and 1½	1.5% CMC in water	0.554	0.920	(*28*)
⊘	⅞ and 1½	3.0% CMC in water	0.566	2.80	(*28*)
⦸	⅞, 1½ and 2	33% Lime water	0.171	0.983	(*28*)
◁	⅞ and 1½	10% Napalm in kerosine	0.520	1.18	(*28*)
▼	8, 10 and 12	4% Paper pulp in water	0.575	6.13	(*1*)
△	¾ and 1½	54.3% Cement rock in water	0.153	0.331	(*31*)
▲	4	18.6% Solids, Mississippi clay in water	0.022	0.105	(*11*)
●	¾ and 1¼	14.3% Clay in water	0.350	0.0344	(*10*)
▷	¾ and 1¼	21.2% Clay in water	0.335	0.0855	(*10*)
×	¾ and 1¼	25.0% Clay in water	0.185	0.204	(*10*)
▽	¾ and 1¼	31.9% Clay in water	0.251	0.414	(*10*)
□	¾ and 1¼	36.8% Clay in water	0.176	1.07	(*10*)
■	¾ and 1¼	40.4% Clay in water	0.132	2.30	(*10*)
▶	⅛, ¼, ½ and 2	23% Lime in water	0.178	1.04	(2)

RESULTS

Figure 1 shows a plot of f vs. $(D^{n'}V^{2-n'}\rho/\gamma$, which has been found to correlate all literature data on non-Newtonian fluids. Actually not all the data points used are shown on the figure; for example, the extensive data of Winding et al. (*32*) were found to correlate excellently, but their fluids were so nearly Newtonian (n' values of 0.885 to 0.985, as compared with 1.000 for a Newtonian fluid) that inclusion of these data would not have contributed significantly. The data of Wilhelm et al. (*31*) for 3-in. pipes were also omitted because of failure of this part of their system to give the correct results with

Newtonian liquids, and similar judgment was used in eliminating the data of many other authors. As it stands, Figure 1 includes data from eight independent investigators, covering sixteen different fluids, the properties of which are tabulated in Table 1. Pipe diameters varied from 1/8 to 12 in. (100-fold variation) and the data cover a 2.1×10^9 range of Reynolds numbers from 6.0×10^{-5} to 130,000. The fluid properties (K' and n') were obtained from rotational viscometers in the case of the *CMC* and lime slurry data of Salt(*24*) and Stevens(*28*) and from capillary-tube or pipe flow data in all other cases. The curves shown in Figure 1 are the commonly accepted relations for Newtonian fluids and the data points represent measurements on non-Newtonians.

Laminar Region. The scattering of the data is significant; although most of the laminar-region data fall with ±10% of the required $f = 16/N_{Re}$ line, several points deviate by 40% or more. As discussed earlier, this scattering must be due to experimental errors or errors in calculation, unless one makes the unlikely assumption that the fluids exhibited thixotropy or rheopexy. Since several of the authors whose data are shown in Figure 1 reported only nominal pipe sizes, it is likely that incorrect values for diameter were occasionally used. Similarily, fluid densities were not always reported accurately. In view of these possible errors the excellence of the correlation within the laminar region is remarkable.

Transition Region. If the previous discussion of the transition from laminar to turbulent flow is indeed valid, then the non-Newtonian data should begin to deviate appreciably from the laminar $f = 16/N_{Re}$ line at approximately the same ratio of viscous shear to inertial forces as do Newtonian data for smooth pipes, namely at $f = 0.008$. This is borne out remarkably well by all the data in this region on Figure 1. Furthermore, because of the close resemblance between highly non-Newtonian fluids at low shear rates and true solids, little eddies of the fluid in turbulent motion should behave more nearly like solid particles and, for example, not break up so readily as do the eddies of Newtonian fluids. The net effect of this difference would be a more diffuse transition from laminar to turbulent flow, mani-

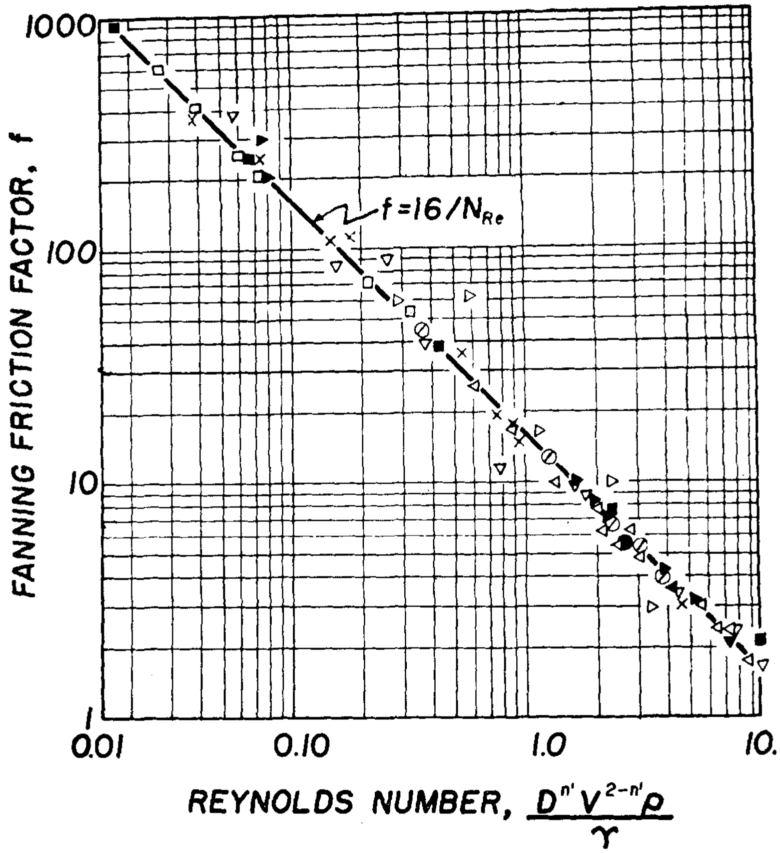

Fig. 1B. Friction-factor–Reynolds-number correlation for non-Newtonian fluids–medium range.

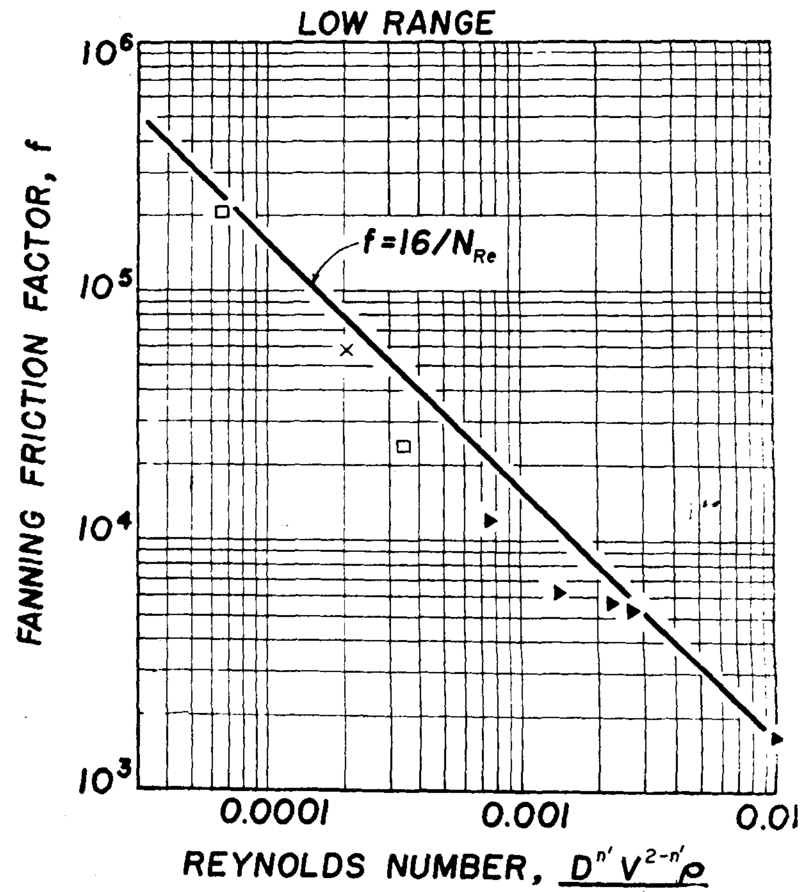

Fig. 1C. Friction-factor–Reynolds-number correlation for non-Newtonian fluids–low range.

fested by a much greater transition region than the 2,000 to 3,300 range of Reynolds numbers found (*14* and *27*) for Newtonian fluids. Whether or not turbulence is fully developed below the maximum Reynolds number of 130,000 (in Figure 1) will have to be shown by further experimental work. Qualitatively, however, the expected diffuse nature and great broadness of the non-Newtonian transition region have been experimentally confirmed by dye-injection studies(*29*) although no work to date has yet defined the actual extent of the transition region for these materials. Until this is done, our tentative approach is to assume that the breadth of the transition region increases as n' decreases and that for most of the fluids whose f–N_{Re} data are shown on Figure 1 the transition region extends approximately to $N_{Re} =$ 50,000 to 70,000. This assumption, although tentative, is supported not only by the data of Figure 1 but also by the data on the less non-Newtonian fluids of Winding et al.(*32*).

It is interesting to compare the proposed criterion for onset of turbulence (i.e., when f first decreases to about 0.008) with the criteria suggested by other investigators. For Newtonian fluids it is obviously identical to the well-established criterion of $N_{Re} =$ 2,000. For Bingham plastic non-Newtonian fluids, Caldwell and Babbitt(*4*) have given an equation which permits the calculation of a "lower critical velocity"—that velocity below which flow will always be laminar—in terms of the pertinent geometric variables and physical properties, which they checked by application to twenty-five different materials. Rearrangement of their equation to enable solving for f gives the identical criterion ($f = 0.008$) proposed in this paper. Hedstrom(*12*) suggested on the basis of extremely limited data that flow of Bingham plastic non-Newtonians becomes turbulent when the various laminar-region curves (for various values of the group $T_y D^2 \rho g_c / \eta^2$ intersect the line relating f and N_{Re} for Newtonian fluids. This is a significantly different criterion from the one proposed in this paper, but inspection of his graphs shows that the experimental data more nearly obey the present criterion than his. Winding et al.(*32*) have presented several criteria: one based on apparent viscosities at zero shear rate, which can obviously not apply to highly non-Newtonian fluids, and one for fluids to which the Williamson equation applies. This latter criterion, when rearranged into the form of a friction factor, again gives as the transition point $f =$ 0.008. The same is also true of other criteria, presented by Toms (*29*) and Metzner(*16*), which make use of Reynolds numbers using apparent viscosities which are so defined as to give the correct pressure drop when substituted in Poiseuille's low. Ooyama and Ito(*18*) presented a criterion which is close to the proposed $f =$ 0.008 but which is hard to compare in greater detail. They supported their criterion experimentally with a small fraction of the data incorporated in the present work; hence it may be concluded that, at least for these data, their criterion gives results identical to those of the more general present one. McMillen(*15*) presented a criterion for Bingham plastic fluids which predicts stable laminar flow at nearly three times the velocities calculated at the point $f = 0.008$. However, he states that his criterion is based on only two experimental measurements, one of which did not leave the laminar region except within the accuracy of the measurements. Accordingly, the discrepancy between his criterion and ours cannot be considered significant. A criterion suggested by Alves et al.(*2*) is too conservative to give the onset of turbulence with accuracy.

In summary, it has been proposed that both Newtonian and non-Newtonian fluids leave the region of stable streamline flow when f first drops to a value of about 0.008 or less or when $D^{n'} V^{2-n'} \rho / \gamma$ reaches a value of 2,000 to 2,500. All available data support this conclusion, and the many different prior-art criteria for onset of turbulence in non-Newtonian fluids which predict it accurately may be rearranged to give the single and perhaps universal criterion proposed here.

Turbulent Region. In view of the uncertainties surrounding the actual width of the transition region, little may be said concerning the difference between the accepted curve for Newtonian fluids in turbulent flow and the experimental non-Newtonian data in the same region of Reynolds numbers. However, if one accepts the foregoing assumption that fully developed turbulence may not occur until $N_{Re} > 50{,}000$ for some fluids, then practically none of the available data fall into what is clearly the region of turbulent flow. Until further work defines the end of the transition region with certainty, it is recommended that the usual Newtonian curve be used for design purposes [with the form of Reynolds number shown in Equation (12)] regardless of the magnitude of N_{Re}. This procedure gives conservative values of pressure drop at $N_{Re} > 2{,}100$, and since the maximum difference between the curve and data in this region is about 50%, Figure 1 is actually very useful even in this "transition" flow region as compared with methods suggested in the prior literature. Winding et al.(*32*) suggested use of the usual f vs. $DV\rho/\mu$ chart together with the evaluation of the viscosity at infinite shear rate, but in large pipes, where turbulence may set in at low velocities, the use of infinite shear-rate viscosities would lead to significant errors unless the fluids are nearly Newtonian in nature. Wilhelm(*31*), Binder(*3*), and Alves (*2*) suggest procedures which require empirical viscosity data in the turbulent region. It would appear that these may be satisfactory design methods if one is able to obtain the necessary data, but they do not permit the design of equipment from physical-properties measurements alone. Other authors(*4*, *10*, *12*, and *18*) have concerned themselves only with Bingham plastic fluids in turbulent flow; hence their work is limited in scope, but the excellent results of Winning(*10*) and Hedstrom(*12*) merit special consideration when fluids of this type are encountered.

COMMENTS ON USE OF CORRELATION

Although the development is perfectly general in that K' and n' were not assumed to be constants (independent of shear rate), it was found that for every fluid on which pipe-line data were available these rheological properties were indeed constant within the accuracy of the data. Had this not been the case, the design procedure to follow would be somewhat more complex; both γ and n' would have to be evaluated at the correct value of shear rate or shear stress. Within the laminar flow region this is not a serious difficulty as γ [Equation (11)] is not a very strong function of n'. Outside the laminar region, however, a trial-and-error procedure would be involved: after calculation of a pressure drop by use of Figure 1, it would be neces-

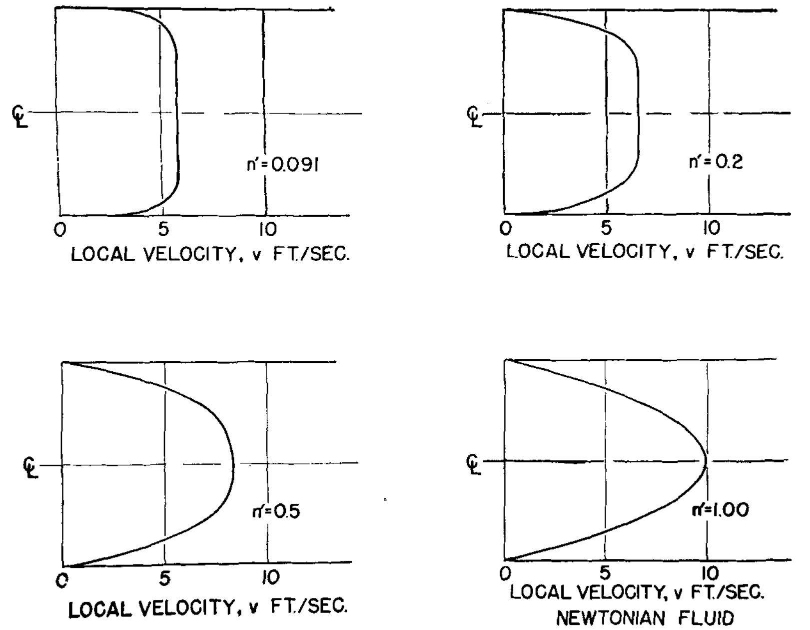

Fig. 2. Dependence of velocity distribution upon the fluid property n'.

sary to calculate T_w and to repeat this procedure until the calculated T_w is identical to that shear stress at which n' and γ were evaluated in obtaining N_{Re}.

Because K' and n' are very nearly constant for so many fluids over shear ranges of practical interest, the logarithmically linear relationship between shear stress and flow rate or shear rate depicted by Equations (5) and (8) constitutes perhaps the most useful means available for extrapolation of rheological data. Theoretical rheologists have long objected to the assumption of the validity of Equations (5) or (8), and Reiner(*23*) has given a particularly good review of their criticisms. Some of these, such as the fact that K' does not have units of viscosity, are irrelevant—N_{Re} as defined by Equation (12) is still rigorously dimensionless. Perhaps the most important apparent objection to the view that almost all fluids obey Equation (5), even when K' and n' are held constant, is the incompatability of this conclusion with the reported existence of materials which behave as Bingham plastics, the "plug flow" of these latter materials in tubes having been experimentally verified by Green (*7, 8* and *9*). However, Figure 2 shows that it is not necessary, as Green(*9*) states, for a material to be a Bingham plastic in order to exhibit plug flow well within experimental accuracy. Indeed, when $n' = 0.20$ it is seen that the velocity profile deviates markedly from the familiar Newtonian parabola and is within 1% of a true plug until r/R becomes greater than 0.4. Seven of the sixteen fluids whose properties are tabulated in Table 1 have values of n' well below 0.20. For these materials it is therefore to be expected that both Equation (5) and the Bingham plastic relationship between shear stress and shear rate apply equally well, a situation that further supports the shelving of classical non-Newtonian fluid definitions and extrapolating, when necessary, by use of Equation (5). The extent to which extrapolation is permissible must, however, be carefully considered in every case, as the rigor of the present treatment depends on the fact that n' and K' are permitted to vary with shear stress. In a sense the variation of these properties with shear stress is similar to the well-known variation of other physical properties with temperature or pressure.

It might appear that the designation of n and K as the true physical properties which describe a non-Newtonian fluid (at a particular shearing stress) would be preferable to the use of n' and K'. The support for this view arises from the fact that the former are independent of the type of apparatus in which the measurements are made, as true physical properties should be, and the latter are not. Actually n and n' are equal numerically for many fluids, and so this choice is not frequently an important one. When they are not equal, the engineering procedure recommended here is to use whichever is more closely related to the problem at hand—for flow in circular pipes this is n'; in other flow situations the use of n will undoubtedly be preferable.

Several important practical conclusions arising from the present work require emphasis. It is instructive to rearrange Equations (9) and (10) to solve for ΔP, whereby one obtains, for flow in the laminar region,

$$\Delta P = \frac{32\,\gamma\,L V^{n'}}{g_c\,D^{n'+1}} \tag{13}$$

For Newtonian fluids ($n' = 1.00$, $\gamma = \mu$) Equation (13) reduces to the usual Poiseuille relationship.

Since $V = 4Q/\pi D^2$,

$$\Delta P = \frac{32\,\gamma\,L}{g_c}\left(\frac{4Q}{\pi}\right)^{n'} \cdot \frac{1}{D^{3n'+1}} \tag{14}$$

If one wishes to reduce the pressure drop accompanying a given volumetric flow rate Q, it is well known that small increases in pipeline diameter are extraordinarily effective for Newtonian fluids as pressure drop varies inversely as diameter to the fourth power when $n' = 1.00$. On the other hand, for highly non-Newtonian systems n' approaches zero, and for such materials it is seen that pressure drop varies inversely as diameter to only the first power; so the designer must go to unusually large pipes before the reduction in pressure drop is appreciable. Increases in capacity of an existing plant, however, may frequently be obtained simply by increasing the pump speed, as the pressure drop is extremely insensitive to flow rate in which n' is small, As a matter of fact, for the slurry data reported by Gregory(*11*) (*see* Table 1) n' is almost zero; i.e., ΔP is very nearly independent of flow rate in the laminar flow region. For engineers familiar only with the peculiarities of Newtonian fluids, these differences are therefore both a hazard and an advantage, but observation of several piping systems installed to date indicates that the industries concerned have taken little advantage of the insensibility of pressure drop to flow rate. These comments must be restricted to the laminar flow region as too few of the present data extend into the region of well-developed turbulent flow to enable one to generalize for that case.

The second practical conclusion depends on the fact that *Figure 1* constitutes a compilation of all the available pipe-line data on highly non-Newtonian systems. As such, it clearly shows that the region of laminar flow is of major interest, as extremely viscous materials are the general rule. However, about one third of the data are outside the laminar region, and this proportion may be expected to increase as turbulence becomes necessary to produce high rates of mass or heat transfer, particularly in nuclear-reactor and similar high-output applications.

FUTURE WORK

The behavior of non-Newtonian fluids in both the transition and turbulent-flow regions should be defined more clearly than has been possible by use of literature data alone. In particular, the extent of the transition region must be clearly defined. A particularly instructive and critical test of of this correlation would be its extension to the turbulent flow of dilatant fluids, on which no data at all were available. Engineering correlations must also be developed at some future date for the more complex thixotropic and rheopetic systems.

SUMMARY

The recommended design procedure, supported by all available data, may be stated as **follows:**

1. *Data Required.* Rheological properties (K' and n') and fluid density are needed. The former should be measured with a capillary-tube viscometer but can frequently be approximated with essentially the same precision from rotational viscometric data. Measurements at only two shear rates (rotational speeds or flow rates) are theoretically sufficient although more are helpful to justify the absence of thixotropy and rheopecty and reduce the experimental errors which frequently tend to be large in this type of work.

2. *Calculations.* γ is obtained from K' and n' [Equation (11)], $N_{Re}(D^{n'}V^{2-n'}\rho/\gamma)$ is calculated, and f is obtained from the usual friction factor —Reynolds number plot. From this point the pressure-drop calculation is identical to the usual procedure for Newtonian fluids. The calculation will be rigorous if the Reynolds number is below 2,100 or if f is greater than 0.008 (laminar flow) and will be conservative in the transition and turbulent-flow regions. Outside the laminar-flow region, the foregoing conclusions may not apply to fluids exhibiting dilatancy.

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NOTATION

Note: As the final correlation is based on dimensionless groups, any consistent set of units may be used. The units given in the following table merely represent those used by the authors.

du/dr = velocity gradient or shear rate, sec.$^{-1}$ $(-du/dr)_w$ refers to the shear rate at the wall of a pipe
D = inside diameter of pipe, ft.
f = Fanning friction factor, defined by Equation (9), dimensionless
g_c = conversion factor, 32.2 ft. lb. mass/(sec.^{2}) (lb.force)
K' = coefficient in Equation (5), (lb.force) (sec.$^{n'}$) / (sq.ft.)
L = length of pipe or tube, ft.
n' = exponent in Equation (5), dimensionless. $n' = 1.000$ for Newtonian fluids, is between zero and unity for pseudoplastic non-Newtonians, and greater than unity for dilatant fluids
N_{Re} = Reynolds number, dimensionless. $N_{Re} = D^{n'}V^{2-n'}\rho/\gamma$ for all fluids, which reduces to $DV\rho/\mu$ for Newtonians
ΔP = pressure drop, lb.force/sq.ft.
Q = volumetric flow rate, cu.ft./sec.
r = distance or radial distance, ft.
R = inside radius of pipe, ft.
T = shear stress, lb.force/sq.ft. T_w denotes shear stress at the wall of a pipe and T_y refers to the yield strength of a Bingham plastic non-Newtonian
v = local velocity, ft./sec.
V = average or bulk velocity, ft./sec.
γ = generalized viscosity coefficient as defined by Equation (11), lb.mass/(ft.) (sec.$^{2-n'}$)
η = coefficient of rigidity of a Bingham plastic non-Newtonian fluid, lb.mass/(sec.) (ft.)
μ = viscosity of a Newtonian fluid, lb.mass/(ft.) (sec.)
ρ = density of fluid, lb.mass/cu.ft.

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