Generalized Differential Quadrature Method for Buckling Analysis

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Abstract: This paper presents a computationally efficient and highly accurate numerical method for analyzing the elastic buckling of columns and plates. The proposed generalized differential quadrature method (GDQM) proposes a simple numerical approach to determine the weighting coefficients for derivative approximations without any restriction on the choice of grid points. It will be shown here that the GDQM is very easy to use and implement numerically. During the solution procedure, different boundary conditions can be easily incorporated. Applications of the GDQM to the buckling analysis of columns and plates have shown that accurate critical buckling loads can be achieved using considerably fewer grid points; thus, less storage and computing time are required during computation. The numerical results obtained, wherever possible, are compared with those from existing literature in order to verify their accuracy.

INTRODUCTION

There is a great deal of research on the buckling analysis of columns and plates. Apart from some simple structure shapes and boundary conditions, numerical methods must be used in the analysis. The commonly used numerical techniques such as the finite-element and finite-difference methods are well developed. However, a large number of grid points are usually required to obtain accurate solutions. Thus, they are computationally expensive.

The differential quadrature method was first introduced by Bellman and Casti (1971) and Bellman et al. (1972) for solving partial differential equations using considerably fewer grid points. This method is based on the idea that the derivative of a function with respect to a coordinate direction can be expressed as a weighted linear sum of the function values at all mesh points along that direction. However, there are some major drawbacks to the original differential quadrature method that restrict its wide applications. These drawbacks are related to the determination of the weighting coefficients for the partial derivative approximation. In the literature, two approaches have been used to obtain the weighting coefficients. One approach is to solve a set of algebraic equations, which satisfy exactly the linear constrained relation for all polynomials of a degree less than or equal to $N - 1$. This set of equations has a unique solution because the matrix elements are composed of a Vandermonde matrix. Unfortunately when the number of grid points N is large, the Vandermonde matrix becomes ill-conditioned and the inversion of this matrix becomes difficult. Moreover, a set of $N \times N$ linear algebraic equations has to be solved for each order derivative even when the equations are solvable. The other approach is to compute the weighting coefficients by a simple algebraic formula, but with the coordinates of grid points chosen as the roots of an Nth-order shifted Legendre polynomials. This means that if N is specified, the distribution of grid points is fixed even for different physical problems or different boundary conditions. This also creates a major drawback and restricts the application of the differential quadrature method, since some practical problems may need more grids near the boundary, while some others may not. Obviously, both methods originally proposed to determine the weighting coefficients in the differential quadrature method have some major drawbacks. This may be one reason why the differential quadrature method is not widely used.

To overcome these drawbacks, a generalized differential quadrature method (GDQM) is introduced here (Shu and Richards 1992; Du et al. 1994). In the GDQM, a simple algebraic formula is obtained to calculate the weighting coefficients of the first-order derivative without any restriction on the choice of the grid points. Further, a recurrence relationship was derived to determine the weighting coefficients for the second- and higher-order derivatives. For the multidimensional cases, each direction can be treated individually, in a manner similar to the one-dimensional (1D) case.

As a result, the GDQM has overcome the possible ill-conditioning problem of the original differential quadrature in obtaining the weighting coefficients. In addition, it avoids solving for the weighting coefficients from a set of algebraic equations. The expressions for the determination of weighting coefficients are so compact and simple that they can be easily implemented in formulating and programming because of their recurrence feature. This method is not only computationally efficient, but also can easily handle various boundary conditions. All these features are convenient, to the generalized differential quadrature, for solving practical problems in structural analysis. Thus, the method is potentially applicable to a wide class of structural problems.

Similar simple algebraic formulas were also obtained by Quan and Chang (1989) to calculate the weighting coefficients of the first- and second-order derivatives. More recently, an important development was published in a paper by Bert et al. (1993). The paper shows that the weighting coefficients for the first-order derivative can be obtained from the differentiation of Lagrangian polynomials, and that a recurrent simple matrix relationship can be used to determine the weighting coefficients for the second- and higher-order derivatives. The GDQM differs from the approach presented by Bert et al. (1993) in two aspects. First, the approach by Bert et al. obtains the weighting coefficients c_{ii} (i.e. $i = j$) for the first-order derivative by directly differentiating the Lagrangian polynomials, while the GDQM presents a straightforward way to calculate these weighting coefficients. Second, the weighting coefficients of higher-order derivatives are calculated by matrix multiplications in the approach presented by Bert et al. (1993), which requires N multiplying operations to obtain each weighting coefficient. In the GDQM introduced here, the determination of the weighting coefficients of higher-order derivatives requires fewer operations as compared to the matrix multiplication approach, especially when N is large.

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In this paper, the proposed GDQM is applied to solve the buckling problems of plates and columns with internal support and various flexural rigidities. It will be shown that accurate results can be obtained by using considerably fewer grid points; thus, it needs considerably less computing effort. The applicability and accuracy of the GDQM are demonstrated throughout the numerical examples.

GENERALIZED DIFFERENTIAL QUADRATURE METHOD

The differential quadrature method (DQM) is based on the idea that the partial derivative of a function, with respect to a space variable at a given discrete point, can be expressed as a weighted linear sum of the function values at all discrete points in the domain of that variable.

Here, a one-dimensional function $u(x, t)$ is taken as an illustrative example. For multidimensional cases, each dimension is treated individually, similar to that in the 1D case. A differential quadrature approximation of the mth-order derivative of the function $u(x, t)$ at the ith discrete point on a grid is given by

$$u_x^{(m)}(x_i, t) = \sum_{j=1}^{N} c_{ij}^{(m)} u(x_j, t) \text{ for } i = 1, 2, \ldots, N, \quad m = 1, 2, \ldots, N-1 \tag{1}$$

where $u_x^{(m)}(x_i, t)$ = mth order derivative of $u(x, t)$ with respect to x at x_i; N = number of discrete grids; and $c_{ij}^{(m)}$ = weighting coefficients for the mth-order derivative approximation.

The most important part of the differential quadrature technique is the determination of the weighting coefficients for any order partial derivative. As mentioned in the introduction, two existing approaches have been used to determine the weighting coefficients in the literature. Both of them have some drawbacks. The first approach may encounter an ill-conditioning problem when the number of grid points becomes large. To quantify this ill-conditioning problem, weighting coefficients have been calculated for equally spaced grids for various numbers of grid points N. Numerical calculations showed that the maximum number of grid points that can be used is $N = 22$. Once the grid number is greater than 22, the set of linear algebraic equations is found to be singular and cannot be solved. The second approach imposes restriction on the choice of the grid points. This leads to a major restriction on this method to problems in structural analysis, since all sorts of boundary conditions could appear and different mesh grids may be needed for different boundary conditions and structure geometry.

In this section, a GDQM is introduced to overcome the aforementioned drawbacks (Shu and Richards 1992; Du et al. 1994). To find a simple algebraic expression to calculate the weighting coefficients without restricting the choice of grid meshes, let us choose the Lagrange interpolated polynomial as the set of test functions $g(x)$

$$g_i(x) = \frac{M(x)}{(x - x_i) \cdot M^{(1)}(x_i)} \text{ for } i = 1, 2, \ldots, N \tag{2}$$

where

$$M(x) = \prod_{j=1}^{N} (x - x_j) \tag{3}$$

N = number of grid points; and $M^{(1)}(x)$ = first derivative of $M(x)$, which is given by

$$M^{(1)}(x_i) = \prod_{j=1, j \neq i}^{N} (x_i - x_j) \tag{4}$$

Let (1) be satisfied by all the test functions $g_i(x)$, then the following simple algebraic expression can be obtained to determine the weighting coefficients for the first-order derivative:

$$c_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j) \cdot M^{(1)}(x_j)} \text{ for } i \neq j \tag{5a}$$

$$c_{ii}^{(1)} = \frac{M^{(2)}(x_i)}{2 \cdot M^{(1)}(x_i)} \text{ for } i, j = 1, 2, \ldots, N \tag{5b}$$

Eq. (5) is a simple expression for computing $c_{ij}^{(1)}$ without any restriction on the choice of the coordinates of grid points x_i. It is obvious that once the grids (x_i) are chosen, $M^{(1)}(x_i)$ can be easily obtained from (4). Hence, $c_{ij}^{(1)}$ can be determined for $i \neq j$. However, for $c_{ii}^{(1)}$, the determination of the coefficients is based on the calculation of the second derivative of $M(x)$, which is slightly more difficult to obtain. Instead of using (5*b*), a more convenient relationship is obtained and used to calculate $c_{ii}^{(1)}$. By using the Taylor series expansion, the following relationship exists for $c_{ij}^{(1)}$:

$$\sum_{j=1}^{N} c_{ij}^{(1)} = 0 \text{ for } i = 1, 2, \ldots, N \tag{6}$$

Thus, from (6), the coefficient $c_{ii}^{(1)}$ can be obtained from $c_{ij}^{(1)}(i \neq j)$. That is

$$c_{ii}^{(1)} = -\sum_{j=1, j \neq i}^{N} c_{ij}^{(1)} \text{ for } i = 1, 2, \ldots, N \tag{7}$$

Similarly, the weighting coefficients for second- and higher-order derivatives can be determined by replacing the test functions in (2) into (1). Consequently, a recurrence relationship has been found for the mth-order weighting coefficients $c_{ij}^{(m)}$

$$c_{ij}^{(m)} = m \cdot \left[c_{ii}^{(m-1)} \cdot c_{ij} - \frac{c_{ij}^{(m-1)}}{x_i - x_j} \right] \text{ for } i \neq j, \; m = 2, 3, \ldots, N-1, \quad i, j = 1, 2, \ldots, N \tag{8}$$

where $c_{ij}^{(m)}$ = weighting coefficients for the mth order derivative, which can be derived from the $(m - 1)$th order weighting coefficients $c_{ij}^{(m-1)}$.

The calculation of $c_{ii}^{(m)}$ can be obtained from the relationship similar to that in (7), that is

$$c_{ii}^{(m)} = -\sum_{j=1, j \neq i}^{N} c_{ij}^{(m)} \text{ for } i = 1, 2, \ldots, N \tag{9}$$

Therefore, (8) and (9), along with (5*a*) and (7), give a convenient and general form to determine the weighting coefficients for the first through $N - 1$th order derivatives. There is no restrictions on the coordinates of the chosen grid points. There is no need to solve for the weighting coefficients from a set of algebraic equations. Further, this set of expressions for the determination of the weighting coefficients is compact and simple and very easy to be implemented in formulating and programming because of its recurrence feature. All these features make it convenient for the generalized differential quadrature for solving practical problems in structural analysis.

Extension of the method to two-dimensional problems is straightforward. Each dimension can be treated individually as a 1D case, assuming there are N_x grid points in the x-direction, $x_1, \ldots, x_{Nx}$, and N_y grid points in the y-direction, $y_1, \ldots, y_{Ny}$. The nth-order partial derivative of $u(x, y)$ with respect to x and the mth-order partial derivative of $u(x, y)$ with respect to y at x_i, y_j can be discretized as

$$u_x^{(n)}(x_i, y_j) = \sum_{k=1}^{N_x} c_{ik}^{(n)} u(x_k, y_j), \; n = 1, \ldots, N_x - 1 \tag{10a}$$

J. Eng. Mech. 1996.122:95-100.

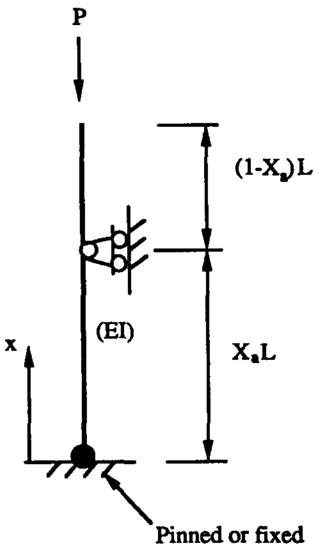

FIG. 1. Column with an Internal Support

TABLE 1. Convergence of GDQM for Columns at $X_a = 1$

Boundary (1)	GDQ ($N = 7$) (2)	GDQ ($N = 9$) (3)	GDQ ($N = 11$) (4)
Pin-pin	10.061	9.8642	9.8697
Fix-pin	19.778	20.255	20.187
P-P [$\overline{EI} = (1 + X)$]	14.478	14.518	14.511
P-P [$\overline{EI} = (1 + X)^2$]	19.709	20.810	20.805
F-P [$\overline{EI} = (1 + X)$]	40.376	29.142	29.441
F-P [$\overline{EI} = (1 + X)^2$]	49.295	43.798	41.968

$$u_y^{(m)}(x_i, y_j) = \sum_{k=1}^{N_y} c_{jk}^{(m)} u(x_i, y_k); \quad m = 1, \ldots, N_y - 1$$

$$\text{for } i = 1, \ldots, N_x, j = 1, \ldots, N_y \tag{10b}$$

The weighting coefficients $c_{ik}^{(n)}$ and $c_{jk}^{(m)}$ can be determined by using (5*a*), (8), and (9) for *x*-direction and *y*-direction discretization, respectively.

APPLICATION OF GDQM TO COLUMN BUCKLING

This section presents the application of the GDQM to column buckling.

The governing equation of a column is given as

$$\frac{d^2}{dX^2}\left(EI\frac{d^2W}{dX^2}\right) = -PL^2\frac{d^2W}{dX^2} \tag{11a}$$

or

$$\overline{EI}\frac{d^4W}{dX^4} + 2\frac{d\overline{EI}}{dX}\frac{d^3W}{dX^3} + \frac{d^2\overline{EI}}{dX^2}\frac{d^2W}{dX^2} = -\frac{PL^2}{EI_0}\frac{d^2W}{dX^2} \tag{11b}$$

where $X = x/L$; $W = w/L$; L = length of the column; and $\overline{EI} = EI/EI_0$ = nondimensionalized flexural rigidity.

On applying the generalized differential quadrature in (1) to (11) at each discrete point on the grids, a set of algebraic equations is obtained:

$$\overline{EI}\sum_{j=1}^{N} c_{ij}^{(4)}\cdot W_j + 2\frac{d\overline{EI}}{dX}\sum_{j=1}^{N} c_{ij}^{(3)}\cdot W_j + \frac{d^2\overline{EI}}{dX^2}\sum_{j=1}^{N} c_{ij}^{(2)}\cdot W_j$$
$$= -\frac{PL^2}{EI_0}\sum_{j=1}^{N} c_{ij}^{(2)}\cdot W_j \tag{12}$$

for $i = 1, 2, \ldots, N$.

It can be written in a matrix form

$$[A]\{W\} = \lambda[B]\{W\} \tag{13}$$

where $\lambda = -PL^2/EI_0$; and $\{W\} = [W_1, W_2, \ldots, W_N]^T$.

The buckling load can be obtained by solving the foregoing eigenvalue problem together with appropriate boundary conditions.

Two different sets of boundary conditions are considered here. The first set is a column with a fixed base and an internal pinned support. The second set is a column with a pinned base and an internal pinned support (Fig. 1). The boundary conditions for each case are as follows:

Fixed base-internal pinned (F-P)

$$W(0) = 0; \quad W'(0) = 0 \tag{14a,b}$$

$$W(X_a) = 0;\ W''(1) = 0;\ EIW''(X_a) = -PL^2W(1) \tag{14c–e}$$

Pinned base-internal pinned (P-P)

$$W(0) = 0; \quad W''(0) = 0 \tag{15a,b}$$

$$W(X_a) = 0;\ W''(1) = 0;\ EIW''(X_a) = -PL^2W(1) \tag{15c–e}$$

where $(\ )'$ = derivative with respect to the spatial coordinate X; and X_a = location of the internal support (Fig. 1).

Using the GDQM, the boundary conditions for each case can be discretized as

1. Fixed-pinned (F-P)

$$W_1 = 0; \quad \sum_{j=1}^{N} c_{1j}^{(1)}\cdot W_j = 0 \tag{16a,b}$$

$$W_{nb} = 0; \sum_{j=1}^{N} c_{N,j}^{(2)}\cdot W_j = 0; \overline{EI}\sum_{j=1}^{N} c_{nj}^{(2)}\ W_j = -\frac{PL^2}{EI_0}W_N \tag{16c–e}$$

2. Pinned-pinned (P-P)

$$W_1 = 0; \quad \sum_{j=1}^{N} c_{1j}^{(2)}\cdot W_j = 0 \tag{17a,b}$$

$$W_{nb} = 0; \sum_{j=1}^{N} c_{N,j}^{(2)}\cdot W_j = 0; \overline{EI}\sum_{j=1}^{N} c_{nj}^{(2)}W_j = -\frac{PL^2}{EI_0}W_N \tag{17c–e}$$

The buckling load for each set of boundary conditions can be obtained by combining the discretized governing (12) with the discretized boundary condition (16) or (17), respectively. To impose the boundary conditions on the governing (12), first (16*a–d*) or (17*a–d*) are used to solve for W_1, W_2, W_{nb}, W_N in terms of the remaining variables $W_3, W_4, \ldots, W_{nb\text{-}1}, W_{nb\text{-}2}, \ldots, W_{N\text{-}1}$. The expressions for W_1, W_2, W_{nb}, W_N in terms of the variables $W_3, W_4, \ldots, W_{nb\text{-}1}, W_{nb\text{-}2}, \ldots, W_{N\text{-}1}$ are then replaced into (12) to eliminate the variables W_1, W_2, W_{nb}, W_N, and only the discretized equations at the points $i = 3, 4, \ldots, nb\text{-}1, nb + 1, \ldots, N - 1$ from (12) are to be used. Eqs. (16*e*) or (17*e*) for each boundary condition case can be taken into account by substituting the equation for the discretized governing equation at the grid point (nb-1) in (12). Finally, the buckling load of columns can be obtained by solving the eigenvalue problem of the remaining $(N - 4) \times (N - 4)$ matrix.

First, the convergence of this method is studied. The buckling loads for a column with the internal support at $X_a = 1$ are calculated for each base support case. For this special internal support location $X_a = 1$, (16*e*) or (17*e*) are no longer needed. Thus, the boundary conditions are relatively simpler. Calculations are performed for columns of various flexural rigidity distributions using a number of grid points. Results obtained are presented in Table 1. The analysis is carried out by using equal space grids. As observed, the convergence of the solution is excellent. Accurate results can be achieved by using

J. Eng. Mech. 1996.122:95-100.

TABLE 2. Convergence of GDQM for Columns at $X_a = 0.5$

N (1)	F-P uniform (2)	F-P $(1 + X)^2$ (3)	P-P uniform (4)	P-P $(1 + X)^2$ (5)
11	9.1349	17.031	4.5741	9.5818
13	5.0223	11.402	4.3017	8.1556
15	3.9961	8.1674	4.4290	8.9427
17	4.6029	10.240	4.3820	8.6866
19	4.4006	9.6574	4.3960	8.7645
21	4.4683	9.8777	4.3895	8.7266
23	4.4636	9.8742	4.3903	8.7286
13 Nonuniform grid	4.4524	9.9114	4.3956	8.6914

TABLE 3. Comparison with Existing Results for Columns at $X_a = 1$

Boundary $\overline{EI}$ (1)	Exact (2)	Reference (3)	GDQ (N = 9) (4)	GDQ (N = 11) (5)
P-P uniform	9.8696 (Jang et al. 1989)	9.9438 (Newberry et al. 1987)	9.8642	9.8697
P-P $(1 + X)$	—	14.3 (Swenson 1952) 15.31 (Bert 1984)	14.518	14.511
P-P $(1 + X)^2$	20.792 (Bleich 1952)	27.455 (Bert 1984)	20.810	20.805
Fix-pin uniform	20.142 (Jang et al. 1989)	20.497 (Newberry 1987)	20.255	20.187

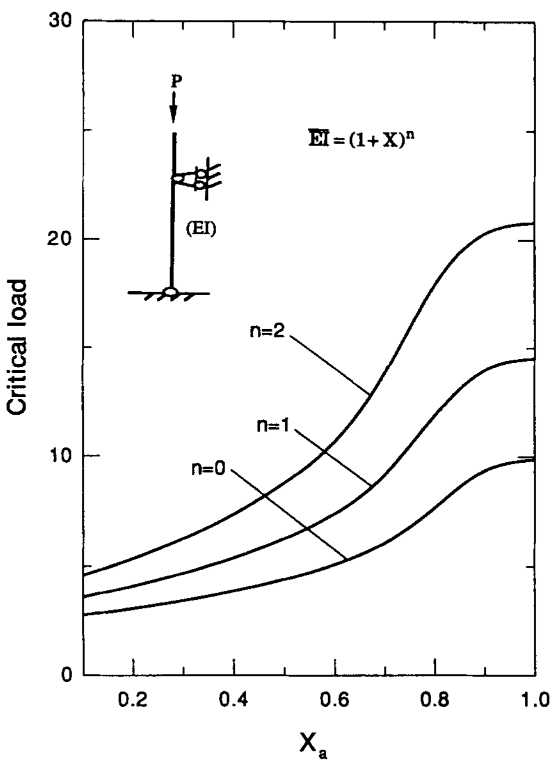

FIG. 2. Critical Loads of Pinned-Base Columns with Internal Support for Various Support Locations X_a and Flexural Rigidities

just nine points, for which only an eigenvalue problem of a 5 × 5 matrix has to be solved. The computing time on an IBM compatible PC-486 for all these cases is less than 1 s.

The second part of the convergence study is to consider a column with the internal support located at $X_a = 0.5$. For this support location, the boundary conditions are more complex due to the added free boundary at the top end (Fig. 1). Again, the buckling loads are computed for each base support case using a number of equal-spaced grid points. The results for both base support cases are tabulated in Table 2. It is obvious that more grid points have to be used to obtain accurate results for columns with internal support due to the added free boundary condition at the top end. Normally, 20 points are required to obtain an accurate solution if equal-spaced grids are used. This implies that an eigenvalue problem of a 16 × 16 matrix has to be solved to extract the accurate buckling loads. The computing time required for this case is less than 2 s on an IBM compatible PC-486. The accuracy can be improved by using nonuniform meshes with finer grids close to the boundaries. The results obtained using 13 nonuniform grid points are also presented in Table 2. The nonuniformly spaced grid points are chosen as X = 0.0, 0.05, 0.1, 0.25, 0.35, 0.45, 0.5, 0.55, 0.6, 0.7, 0.9, 0.95, 1.0. Using only 13 points, we can obtain very good results. The computing time on an IBM compatible PC-486 is less than 1 s because we only have to solve an eigenvalue problem for a 9 × 9 matrix after imposing the boundary conditions.

To demonstrate the accuracy of this method, results are obtained for columns with a support located at $X_a = 1$, for which some analytical and approximate solutions are available for comparison. The results are presented in Table 3 along with the analytical and approximate solutions available in the literature. The comparison shows that accurate results can be achieved by using considerably fewer grid points. For all the cases shown, very accurate results can be achieved by using just nine points. This means we only need to solve the eigenvalue problem of a 5 × 5 matrix. The computing time for all these cases is less than 1 s on an IBM compatible PC-486. Obviously, good accuracy can be achieved with a very small computing effort.

Extensive calculations are finally performed to show the effect of the location of the internal support X_a on buckling loads for various columns. The first one is for columns with a fixed base. The second one is for columns with a pinned base. The location of the internal support is varied from 0.1 to 1. Three

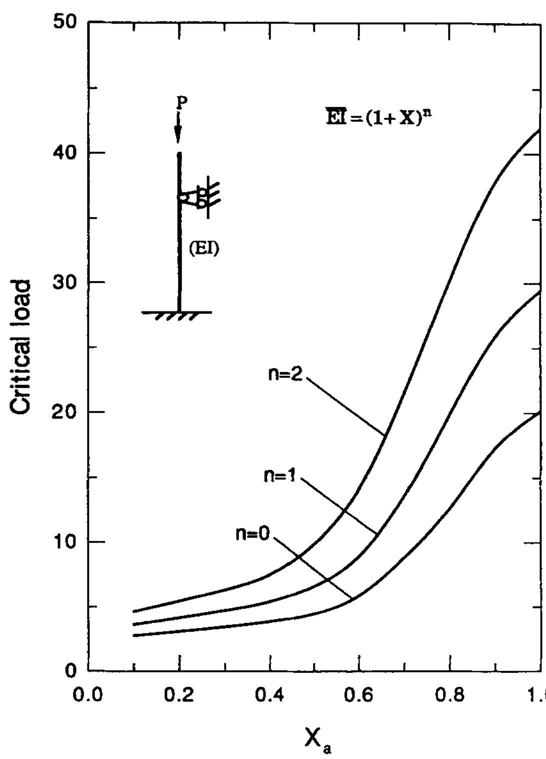

FIG. 3. Critical Loads of Fixed-Base Columns with Internal Support for Various Support Locations X_a and Flexural Rigidities

J. Eng. Mech. 1996.122:95-100.

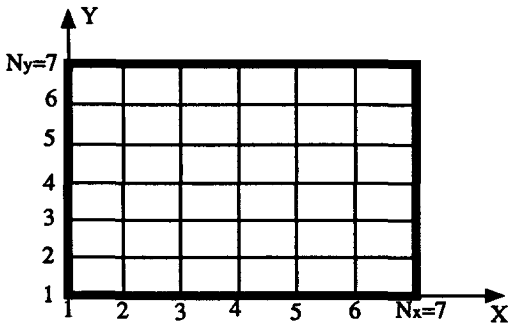

FIG. 4. Grid of Rectangular Plate

TABLE 4. Buckling Loads of Square Plates

Boundary (1)	Exact (2)	7 × 7 (3)	Error (%) (4)	9 × 9 (5)	Error (%) (6)	11 × 11 (7)	Error (%) (8)
S-S-S-S	39.4784	39.8652	0.98	39.4678	0.027	39.4786	0.00
C-C-C-C	99.3869	94.2733	5.1	97.1653	2.2	99.8377	0.45

TABLE 5. Comparison of Buckling Loads from FEM and GDQ

FEM mesh (1)	6 × 6 (2)	8 × 8 (3)	10 × 10 (4)	Exact (5)
FEM result	38.3633	38.8173	39.0838	39.4784
GDQ grid	7 × 7	9 × 9	11 × 11	—
GDQ result	39.8652	39.4678	39.4786	—

types of flexural rigidity distribution $\overline{EI}$ of columns are considered, i.e., $\overline{EI} = (1 + X)^n$, where $n = 0$, 1, and 2. The results are presented in Figs. 2 and 3 for pinned-base and fixed-base columns, respectively. The results show that the critical load value increases significantly with the increase of X_a and flexural rigidity $\overline{EI}$.

APPLICATION OF GDQM TO BUCKLING OF PLATES

The governing equation for the buckling of a thin rectangular plate under uniaxial load N_x is given as

$$D\frac{\partial^4 w}{\partial x^4} + 2D\frac{\partial^4 w}{\partial x^2 \partial y^2} + D\frac{\partial^4 w}{\partial y^4} = N_x\frac{\partial^2 w}{\partial x^2} \tag{18}$$

where D = flexural rigidity of the plate.

On normalizing (18), it becomes

$$\frac{\partial^4 W}{\partial X^4} + 2\beta^2\frac{\partial^4 W}{\partial X^2 \partial Y^2} + \beta^4\frac{\partial^4 W}{\partial Y^4} = \frac{N_x a^2}{D}\frac{\partial^2 W}{\partial X^2} \tag{19}$$

where $\beta = a/b$; $X = x/a$; $Y = y/b$; a = length of the plate; and b = width of the plate.

By applying the generalized differential quadrature approximation (10) to (19) at each discrete point on the grid (Fig. 4), we have

$$\sum_{k=1}^{N_x} c_{ik}^{(4)} W_{kj} + 2\beta^2 \sum_{m=1}^{N_y} c_{jm}^{(2)} \sum_{k=1}^{N_x} c_{ik}^{(2)} W_{km} + \beta^4 \sum_{k=1}^{N_y} c_{jk}^{(4)} W_{ik}$$

$$= \frac{N_x a^2}{D} \sum_{k=1}^{N_x} c_{ik}^{(2)} W_{kj} \quad i = 1, 2, \ldots, N_x, j = 1, 2, \ldots, N_y \tag{20}$$

where N_x, N_y = number of grid points along the X-direction and Y-direction, respectively (Fig. 4).

The boundary conditions for a plate clamped on all four edges (C-C-C-C) are

$$W(X, 0) = W(X, 1) = W(0, Y) = W(1, Y) = 0 \tag{21a}$$

$$\frac{\partial W}{\partial Y}(X, 0) = \frac{\partial W}{\partial Y}(X, 1) = \frac{\partial W}{\partial X}(0, Y) = \frac{\partial W}{\partial X}(1, Y) = 0 \tag{21b}$$

Applying GDQ (10) to the boundary conditions (21)

$$W_{1j} = W_{Nj} = W_{i1} = W_{iN} = 0 \tag{22a}$$

$$\sum_{k=1}^{N_x} c_{1k}^{(1)} W_{kj} = \sum_{k=1}^{N_x} c_{Nk}^{(1)} W_{kj} = \sum_{k=1}^{N_y} c_{1k}^{(1)} W_{ik} = \sum_{k=1}^{N_y} c_{Nk}^{(1)} W_{ik} = 0 \tag{22b}$$

for $i = 1, 2, \ldots, N_x$ and $j = 2, 3, \ldots, N_y - 1$.

For a plate with all four edges simply supported (S-S-S-S), the boundary conditions are

$$W(X, 0) = W(X, 1) = W(0, Y) = W(1, Y) = 0 \tag{23a}$$

$$\frac{\partial^2 W}{\partial Y^2}(X, 0) = \frac{\partial^2 W}{\partial Y^2}(X, 1) = \frac{\partial^2 W}{\partial X^2}(0, Y) = \frac{\partial^2 W}{\partial X^2}(1, Y) = 0 \tag{23b}$$

Applying GDQ (10) to the boundary conditions (23)

$$W_{1j} = W_{Nj} = W_{i1} = W_{iN} = 0 \tag{24a}$$

$$\sum_{k=1}^{N_x} c_{1k}^{(2)} W_{kj} = \sum_{k=1}^{N_x} c_{Nk}^{(2)} W_{kj} = \sum_{k=1}^{N_y} c_{1k}^{(2)} W_{ik} = \sum_{k=1}^{N_y} c_{Nk}^{(2)} W_{ik} = 0 \tag{24b}$$

for $i = 1, 2, \ldots, N_x$ and $j = 2, 3, \ldots, N_y - 1$.

Similarly, the buckling loads for rectangular plates can be obtained by solving the eigenvalue problem in (20) together with the appropriate boundary conditions in (22) or (24).

In the present analysis for rectangular plates, two different sets of boundary conditions, with four edges clamped and four edges simply supported, are considered. Numerical results are obtained using a number of grid points to study the convergence of the solution. The results are presented in Table 4 along with exact values for comparison. Good convergence of the solutions are observed from the table. For the simply supported plate (S-S-S-S), 7 × 7 grid points can produce quite accurate results. The results are obtained by solving an eigenvalue problem of a 9 × 9 matrix, for which the computing time on a PC-486 is less than 1 s. For the clamped plate (C-C-C-C), good results can be achieved by using 9 × 9 grid points. This requires a solution of an eigenvalue problem of a 25 × 25 matrix. The computing time on a PC-486 is less than 2.5 s for this case.

Buckling loads for simply supported square plates, obtained through the finite-element method and the generalized differential quadrature method, are also presented in Table 5 for comparison. The finite-element solutions from Zienkiewicz (1977) and Allen and Bulson (1980) are obtained by using a rectangular plate element. As observed, good result can be obtained by using a 10 × 10 mesh in the finite-element method. For the 10 × 10 mesh, the dimension of the resulting system matrix is 243 × 243 for plates with four edges simply supported. For the 6 × 6 mesh, the dimension of the resulting system matrix is 75 × 75 for plates with four edges simply supported. For the 6 × 6 mesh and 8 × 8 mesh shown in the table, the dimension of the resulting system matrix are 75 × 75 and 147 × 147, respectively. On the other hand, accurate results can be achieved by using 9 × 9 grid points in the GDQ. The dimension of the resulting system matrix in the GDQ solution is only 25 × 25 for this case. The required computing time on a PC-486 is less than 2.5 s. The dimension of the resulting system matrix for 7 × 7 grid points is only 9 × 9. The required computing time is less than 1 s.

CONCLUSIONS

In this paper, a generalized differential quadrature method was introduced to study the column buckling with an internal support and varying flexural rigidity and the buckling prob-

J. Eng. Mech. 1996.122:95-100.

lems of rectangular plates. This method proposes a very simple algebraic formula to determine the weighting coefficients required by the differential quadrature approximation without restricting the choice of mesh grids. Numerical calculations showed that accurate results can be achieved using considerably fewer grid points and much less storage and computing time. The solution procedures and programming are very simple and easy. In addition, boundary conditions are easily incorporated into the solution procedure. It can be concluded that due to its superb accuracy, efficiency, and convenience, GDQM has great potential for wide use in structural analysis.

APPENDIX I. REFERENCES

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APPENDIX II. NOTATION

The following symbols are used in this paper:

$c_{ij}^{(m)}$ = weighting coefficient for the mth order derivative approximation;
EI = flexural rigidity of column;
EI_0 = reference flexural rigidity of column;
$\overline{EI}_0$ = nondimensionalized flexural rigidity;
L = length of column;
N = number of grid points;
P = axial load acting at the end of column;
$u(x, t)$ = a one dimensional function;
$u_x^{(m)}(x, t)$ = mth order derivative of $u(x, t)$ with respect to x;
W = nondimensionalized transverse deflection;
w = transverse deflection of column;
X = nondimensionalized coordinate;
X_a = location of internal support of column; and
x = coordinate system.

J. Eng. Mech. 1996.122:95-100.