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Critical load for buckling of non-prismatic columns under self-weight and tip force

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ABSTRACT

The stability of elastic columns with variable cross-section under self-weight and concentrated end load is considered. A simple and easy-to-implement approach is suggested. Different end conditions are dealt with. The governing equation subject to associated boundary conditions for Euler–Bernoulli columns is transformed into an integral equation, and critical buckling load is then evaluated by seeking the lowest eigenvalue of the resulting integral equation. Numerical examples of the critical buckling load for prismatic and non-prismatic columns under self-weight and end force are given, and the effectiveness of this method for buckling analysis of tapered columns is validated. For several frequently encountered end supports, the influence of the taper ratio on the critical buckling load is discussed.

1. Introduction

Elastic columns are a class of important structural elements, which have been widely used in civil, mechanical, and aerospace engineering. The determination of critical load for buckling of elastic columns is a key problem in engineering design. The first researcher in this field can be traced back to Euler, who pioneered the study of buckling of a prismatic column subjected to a compressive force or under its own weight. Since then, great progress in this field has been made. For example, Gere and Carter (1962) derived exact buckling solutions for several special types of tapered columns with simple boundary conditions in terms of Bessel functions. Siginer (1992) also used the Bessel functions to deal with the buckling of columns with linearly varying inverse of the bending stiffness. For parabolically varying bending stiffness, Ermopoulos (1986) studied the buckling of tapered columns subjected to axially concentrated loads at any position along the length direction. Williams and Aston (1989) further analyzed bounds of the buckling load of tapered columns with certain special second moment of area. With the aid of the Bessel functions, Li (2001) gave a variety of exact solutions for buckling of non-uniform columns under axial concentrated and distributed loading. Using the inverse method, Elishakoff (1999, 2000, 2001) obtained several closed-form solutions for the buckling of inhomogeneous columns with special variable bending stiffness. Furthermore, Li (2009) employed the inverse method and gave exact solutions for the generalized Euler's problem. For a prismatic column under self-weight and tip force, Duan and Wang (2008) exactly determined the buckling load in terms of generalized hypergeometric functions. Recently, Darbandi et al. (2010) put forward to the perturbation method to determine the buckling load of columns with variable cross-section under axial loading. Wang (2010) investigated the stability of a braced standing heavy column and obtained an optimum location of the support for maximum load-carrying capacity. Huang and Li (in press) dealt with buckling of axially graded columns with any axial nonhomogeneity and further gave a suboptimal design of the shape profile of a homogeneous column with constant weight constraint.

Although many methods have been presented to solve the stability of elastic columns with variable cross-section under different boundary conditions, most of them have strict limitation. For instance, application of the special function method such as using Bessel functions strongly depends on the form of an ordinary differential equation with variable coefficients. This paper presents a procedure for determining the buckling load of prismatic and non-prismatic columns under self-weight and tip force.

2. Basic equation

Consider the buckling of a non-prismatic elastic column of length L subjected to an axial compressive force P at its upper tip. When the effect of its own weight is taken into account, we denote distributed axial load as $Q(x)$ along its length direction to describe this effect, where x stands for the axial coordinate measured from the bottom end (Fig. 1). Under such a circumstance, the govern-

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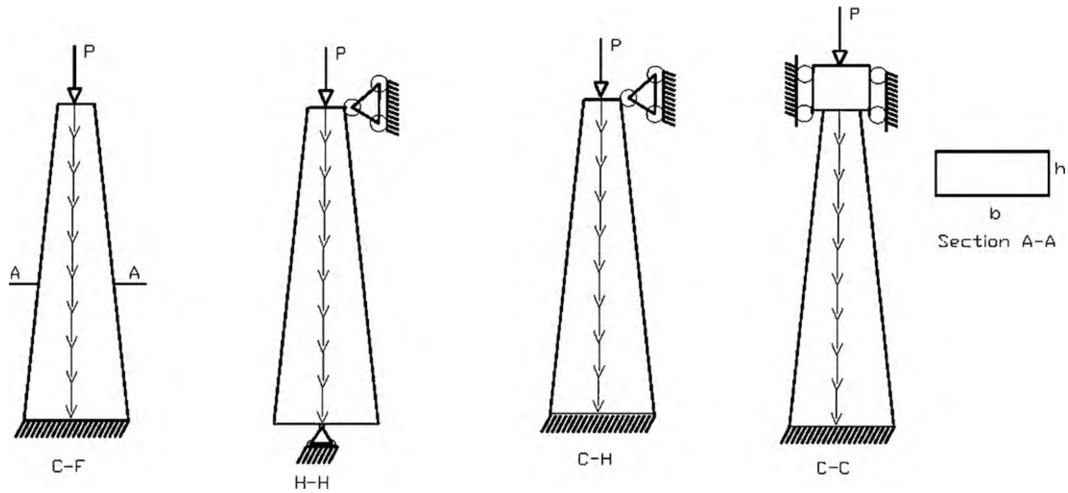

Fig. 1. Schematic of non-prismatic columns under self-weight and tip force.

ing differential equation for elastic buckling of Euler–Bernoulli columns is

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2w}{dx^2}\right]+\frac{d}{dx}\left[P+Q(x)\frac{dw}{dx}\right]=0,\quad 0<x<L, \tag{1}$$

where E is Young's modulus, w is the deflection, and $I(x)$ is the area moment of inertia.

To simplify our analysis, in what follows we introduce the following dimensionless variable: $\xi = x/L$, and consider a non-prismatic column with rectangular cross-section of linearly varying width and thickness, say. Then we have

$$EI(x)=EI_0 I(\xi),\qquad Q(x)=q_0 L f(\xi) \tag{2}$$

with

$$I(\xi)=(1-\alpha_1\xi)(1-\alpha_2\xi)^3,\qquad f(\xi)=\int_\xi^1 (1-\alpha_1\xi)(1-\alpha_2\xi)\,d\xi, \tag{3}$$

where α_1, α_2 are two taper ratios with respect to the width and thickness directions, respectively, and q_0 denotes weight per unit length. Moreover, $0 \le \alpha_j \le 1$. In particular, when $\alpha_j = 1$, the rectangular column tapers to a sharp tip. This case may make the critical buckling load of elastic columns vanish (Timoshenko and Gere, 1961), and is nearly a theoretical limit because it can never be reached in practice. Consequently, Eq. (1) can be transformed into a normalized form

$$\frac{d^2}{d\xi^2}\left[I(\xi)\frac{d^2w}{d\xi^2}\right]+\lambda_p\frac{d^2w}{d\xi^2}+\lambda_q\frac{d}{d\xi}\left[f(\xi)\frac{dw}{d\xi}\right]=0, \tag{4}$$

where

$$\lambda_p=\frac{PL^2}{EI_0},\qquad \lambda_q=\frac{q_0L^3}{EI_0}. \tag{5}$$

Now we integrate both sides of Eq. (4) four times and then get

$$I(\xi)w(\xi)+\int_0^\xi K(\xi,s)w(s)\,ds=\frac{C_1}{6}\xi^3+\frac{C_2}{2}\xi^2+C_3\xi+C_4 \tag{6}$$

where C_j (j = 1, 2, 3, 4) are unknown constants to be determined through boundary conditions at both ends, and the kernel function $K(\xi, s)$ is

$$K(\xi,s)=-2I'(s)+(\xi-s)[I''(s)+\lambda_p+\lambda_q f(s)]-\frac{\lambda_q}{2}(\xi-s)^2 f'(s). \tag{7}$$

In the above, the prime represents differentiation with respect to the argument.

3. Solution procedure

Here consider several typical heavy columns under an end force. For clamped-free (C-F) columns with clamped end x = 0 and free end x = L, the boundary conditions for this case can be written below

$$w(0)=0,\qquad w'(0)=0, \tag{8}$$

$$\frac{d^2w}{d\xi^2}=0,\qquad \frac{d}{d\xi}\left[I(\xi)\frac{d^2w}{d\xi^2}\right]+\lambda_p\frac{dw}{d\xi}=0,\qquad \xi=1. \tag{9}$$

Using the above conditions, we easily find

$$C_1=0,\quad C_2=\frac{-2}{2I(1)-\lambda_p}\int_0^1[\lambda_p K(1,s)+\lambda_q I(1) f'(s)]w(s)\,ds.$$
$$C_3=0,\quad C_4=0$$

After substitution of the above results of C_j into Eq. (6), we get an integral equation with respect to w as follows:

$$I(\xi)w(\xi)+\int_0^\xi K(\xi,s)w(s)\,ds+\int_0^1 H(\xi,s)w(s)\,ds=0, \tag{10}$$

where

$$H(\xi,s)=\frac{\xi^2}{2I(1)-\lambda_p}[\lambda_p K(1,s)+\lambda_q I(1)f'(s)]=0. \tag{11}$$

Using the same procedure, for other elastic columns with the following boundary conditions:

$w=0,\quad w'=0$ at $\xi=0,1,$

for clamped-clamped (C-C) columns;

$w=0,\quad w''=0$ at $\xi=0,1,$ for hinged-hinged (H-H) columns;

$w(0)=w'(0)=0,\quad w(1)=w''(1)=0,$

for clamped-hinged (C-H) columns;

we similarly obtain an integral equation for each case, which is still expressed by (10), but with the following kernel

$$H(\xi,s)=\begin{cases}-\xi^3[K'_\xi(1,s)-2K(1,s)] & \\ -\xi^2[3K(1,s)-K'_\xi(1,s)] & \text{for C-C columns,}\\ \frac{1}{6}\xi^3\lambda_q f'(s)-\xi[K(1,s)+\frac{\lambda_q}{6}f'(s)] & \text{for H-H columns,}\\ \frac{\xi^3-\xi^2}{4}\lambda_q f'(s)+\frac{\xi^3-3\xi^2}{2}K(1,s) & \text{for C-H columns.}\end{cases} \tag{12}$$

where $K'_\xi(\xi,s)=\partial K(\xi,s)/\partial\xi$.

As a consequence, it suffices to determine the eigenvalues of the resulting integral Eq. (10). The critical buckling load is in fact related

Table 1
Critical load λ_p for a column with constant width and linearly varying thickness ($\alpha_1 = 0$).

α_2	C-F			H-H			C-H			C-C
	Present	SPM	FEM	Present	SPM	FEM	Present	SPM	FEM	Present
0	2.467	2.47	2.47	9.870	9.87	9.87	20.191	20.19	20.19	39.478
0.2	2.023	2.02	2.02	7.090	7.09	7.09	14.494	14.49	14.5	28.330
0.4	1.569	1.57	1.57	4.685	4.69	4.69	9.543	9.54	9.55	18.626
0.6	1.098	1.10	1.10	2.672	2.67	2.68	5.388	5.39	5.40	10.479
0.8	0.597	0.61	0.60	1.082	1.09	1.09	2.119	2.12	2.13	4.084

Results in SPM and FEM columns are taken from Darbandi et al. (2010).

to the lowest eigenvalue. To this end, the resulting integral equation is approximated by a system of linear equations. We expand $w(\xi)$ as power series in ξ. More precisely, if neglecting sufficiently small error, the unknown $w(\xi)$ can be approximately represented as $w_N(\xi)$:

$$w_N(\xi) = \sum_{n=0}^{N} c_n \xi^n \tag{13}$$

where c_n are unknown coefficients and N is a certain positive integer, which is chosen large enough such that the rest terms have a negligible error. Clearly, when N is chosen larger and larger, this approximation $w_N(\xi)$ approaches the desired solution $w(\xi)$. When N is raised, a stable eigenvalue can be easily reached (Huang and Li, 2010). In the following evaluation, $N = 10$ is assumed. Note that some more effective expansions may be orthogonal polynomials such as Chebyshev and Legendre polynomials (Atkinson and Han, 2005). However, thanks to the simplicity of the power series and the effectiveness of the method, here we utilize (13) to approximate $w(\xi)$.

Next we substitute (13) into (10), yielding

$$\sum_{n=0}^{N} c_n \xi^n I(\xi) + \sum_{n=0}^{N} c_n \int_0^{\xi} K(\xi, s) s^n \, ds + \sum_{n=0}^{N} c_n \int_0^{1} H(\xi, s) s^n \, ds = 0. \tag{14}$$

Furthermore, we multiply both sides of Eq. (14) by ξ^m and then integrate with respect to x between 0 and 1, yielding a system of linear algebraic equations in c_n:

$$\sum_{n=0}^{N} (i_{mn} + k_{mn} + h_{mn}) c_n = 0; \quad m = 0, 1, 2, \ldots, N \tag{15}$$

where

$$i_{mn} = \int_0^1 \xi^{m+n} I(\xi) \, d\xi, \qquad k_{mn} = \int_0^1 \int_0^{\xi} K(\xi, s) \xi^m s^n \, ds \, d\xi,$$
$$h_{mn} = \int_0^1 \int_0^1 H(\xi, s) \xi^m s^n \, ds \, d\xi. \tag{16}$$

To obtain a nontrival solution of the resulting homogeneous system, the determinant of the coefficient matrix of the system (15) has to vanish. Accordingly, we obtain a characteristic equation in λ_p or λ_q, from which one can readily obtain the desired eigenvalues. Among these obtained eigenvalues, the lowest one corresponds to the critical buckling load. In general, for $I(\xi)$, $K(\xi, s)$, $H(\xi, s)$ in (16) being polynomials with respect to the argument(s), the coefficient matrix in (15) can be analytically calculated. However, for complicated expressions for $I(\xi)$, $K(\xi, s)$, $H(\xi, s)$, we can obtain numerical results by performing numerical computation. Using integration evaluation in the library function in Matlab, we get the matrix coefficients and derive final results.

4. Numerical examples

In this section, let us consider several numerical examples to validate the efficiency of the suggested method. The first one is devoted to a non-prismatic column under a concentrated compressive force, the second is concerned with the buckling of a non-prismatic column under its own weight, and the last one analyzes the buckling of a non-prismatic column subjected to its own weight and tip force simultaneously.

Example 1. Non-prismatic columns under end force.

Consider a non-prismatic elastic column under a compressive force P at the end $x = L$. In this example, the weight of the column is neglected, and then $q_0 = 0$. Here we study three different cases of cross-section. One corresponds to constant width and linearly varying thickness, which means $\alpha_1 = 0$ and $I(\xi) = (1 - \alpha_2 \xi)^3$. Numerical results of the critical buckling load λ_p are calculated and they are listed in Table 1. These results agree very well with those by using the finite-element method (FEM) and the singular perturbation method (SPM) (Darbandi et al., 2010). It is worth noting that the results corresponding to clamped-clamped columns were not given in Darbandi et al. (2010).

For an elastic column with constant thickness and linearly varying width, one has $\alpha_2 = 0$ and $I(\xi) = 1 - \alpha_1 \xi$. Numerical results are also evaluated under the corresponding boundary conditions. To compare our results with those derived from the finite-element method, we also compute the latter for a steel Q235 column with $E = 210.06$ GPa, $\rho = 8.01$ g/cm^3. Its cross-section at $\xi = 0$ is a rectangle of 3 m × 2 m and the length of the column is taken 75 m. Using commercial software Midas/Civil V. 6.7.1 we take 10 elements with cross-section and axial force piecewise constant, and the evaluated numerical results together with the above-obtained ones are presented in Table 2. From Table 2, one finds that obtained results are quite close and relative error is lower than 1%.

By comparing Table 2 with Table 1, we can see that the critical buckling loads of Example 2 are greater than the corresponding values of Example 1 for the same taper ratio with the same end supports. This implies that an elastic column with constant thickness and linearly varying width (Example 2) has a stronger load-carrying capacity than that with constant width and linearly varying thickness. In other words, an elastic column under an axial compressive force is easy to buckle towards the linearly varying thickness direction, rather than the linearly varying width direc-

Table 2
Critical load λ_p for a column with constant thickness and linearly varying width ($\alpha_2 = 0$).

α_1	C-F		H-H		C-H		C-C	
	Present	FEM	Present	FEM	Present	FEM	Present	FEM
0	2.467	2.47	9.870	9.85	20.191	20.11	39.478	39.21
0.2	2.316	2.31	8.864	8.85	18.119	18.04	35.412	35.17
0.4	2.151	2.15	7.809	7.79	15.912	15.85	31.043	30.83
0.6	1.968	1.97	6.679	6.67	13.502	13.44	26.198	26.04
0.8	1.752	1.75	5.411	5.40	10.720	10.67	20.427	20.27

Table 3
Critical load λ_p for a column with square cross-section ($\alpha_1 = \alpha_2 = \alpha$).

α_1	C-F		H-H		C-H		C-C	
	Present	FEM	Present	FEM	Present	FEM	Present	FEM
0	2.467	2.47	9.870	9.85	20.191	20.11	39.478	39.21
0.2	1.884	1.88	6.317	6.31	12.922	12.89	25.266	25.14
0.4	1.309	1.31	3.553	3.56	7.269	7.27	14.212	14.19
0.6	0.757	0.76	1.579	1.60	3.230	2.36	6.316	6.36
0.8	0.265	0.28	0.395	0.42	0.807	0.86	1.547	1.64

tion. Such a conclusion is easily understood since the bending stiffness $EI_0(1-\alpha\xi)^3$ is less than $EI_0(1-\alpha\xi)$. Therefore, the elastic column tends to deflect towards the side of constant width and linearly varying thickness.

The third case is devoted to an elastic column with square cross-section. Note that a column with circular cross-section can be similarly analyzed and omitted here. In this case, $\alpha_1 = \alpha_2 = \alpha$ and $I(\xi) = (1-\alpha\xi)^4$. Table 3 shows the numerical results of critical buckling load λ_p. In particular, when $\alpha = 0$, our results for the four cases of end support are identical to the well-known Euler's results. Here finite-element results are also given in Table 3.

Example 2. Non-prismatic columns under self-weight.

Now we consider the buckling of a non-prismatic elastic column only under its own weight. This indicates $P = 0$ and the critical load parameter is characterized by λ_q. For an elastic column with constant width and linearly varying thickness, we have $\alpha_1 = 0$, $I(\xi) = (1-\alpha_2\xi)^3$, $f(\xi) = \int_\xi^1 (1-\alpha_2 s)\,ds$. The critical buckling load λ_q is calculated under various boundary conditions and numerical results are presented in Table 4. For comparison, the previous results for a prismatic column in Elishakoff (2000) and in Duan and Wang (2008) are also given in Table 4. Excellent agreement of our results with those in Elishakoff (2000) and in Duan and Wang (2008) is observed in the case of $\alpha_2 = 0$.

In addition, one further has $\alpha_2 = 0$, $I(\xi) = 1-\alpha_1\xi$, $f(\xi) = \int_\xi^1 (1-\alpha_1 s)\,ds$ for a column with constant thickness and linearly varying width, or $\alpha_1 = \alpha_2 = \alpha$, $I(\xi) = (1-\alpha\xi)^4$, $f(\xi) = \int_\xi^1 (1-\alpha s)^2\,ds$ for a column with varying square cross-section, respectively. The numerical results of the critical buckling load λ_q are given in Tables 5 and 6. It is interesting to note that the critical buckling load λ_q under self-weight increases with the increment of α_1 or α_2 for clamped-free columns. Nevertheless, the dependence of λ_q on other end supports is reversed for either $\alpha_1 = 0$ or $\alpha_2 = 0$. So we can come to the conclusion that an elastic column is prone to buckle towards the linearly varying thickness direction.

Example 3. Non-prismatic columns under self-weight and end force.

Table 4
Critical load λ_q for a column with constant width and linearly varying thickness ($\alpha_1 = 0$).

α_2	C-F	H-H	C-H	C-C
0	7.837	18.569	52.501	74.629
	(7.837)	(18.6)	(52.5)	(74.6)
	[7.8373]	[18.5687]	[52.5007]	[74.6286]
0.2	8.076	16.237	48.072	66.964
0.4	8.412	13.726	43.327	58.781
0.6	8.932	10.927	38.073	49.783
0.8	9.911	7.564	31.784	39.178
1	13.187	0	13.187	13.187

Results in parenthesis and square bracket are taken from Elishakoff (2000) and Duan and Wang (2008), respectively.

Table 5
Critical load λ_q for a column with constant thickness and linearly varying width ($\alpha_2 = 0$).

α_1	C-F	H-H	C-H	C-C
0	7.837	18.569	52.501	74.629
0.2	8.853	19.385	56.221	79.187
0.4	10.301	20.393	61.119	84.978
0.6	12.554	21.669	67.941	92.591
0.8	16.592	23.307	78.262	102.796

Table 6
Critical load λ_q for a column with square cross-section ($\alpha_1 = \alpha_2 = \alpha$).

α	C-F	H-H	C-H	C-C
0	7.837	18.569	52.501	74.629
0.2	9.089	16.817	51.315	70.889
0.4	10.856	14.467	49.675	66.215
0.6	13.584	11.204	47.213	60.025
0.8	18.527	6.535	42.915	50.878
1	30.530	0	30.530	30.530

Table 7
Critical load λ_p for a prismatic column under self-weight.

$4\lambda q/\pi^2$	C-F	H-H	C-H	C-C
−10	8.771	21.183	28.342	51.576
−5	5.849	15.778	24.360	45.587
−3	4.561	13.477	22.716	43.158
−2	3.886	12.295	21.882	41.936
−1	3.189	11.093	21.040	40.710
0	2.467	9.870	20.191	39.478
0.5	2.097	9.250	19.763	38.861
1	1.721	8.625	19.333	38.242
2	0.947	7.360	18.467	37.001
3	0.145	6.075	17.593	35.756
5	−1.552	3.441	15.818	33.250
10	−6.377	−3.491	11.227	26.902

Finally, consider a non-prismatic column in practical application when self-weight and end force are simultaneously present. We calculate the critical buckling load λ_p of a prismatic and non-prismatic column for prescribed distributed axial load λ_q. For a prismatic column, the width and thickness are constant, i.e. $I(\xi) = 1$, $f(\xi) = 1$, and obtained results are shown in Table 7. For a non-prismatic column, for the sake of simplicity we only consider the case of square cross-section. Moreover, we take $\alpha_1 = \alpha_2 = 0.8$ and

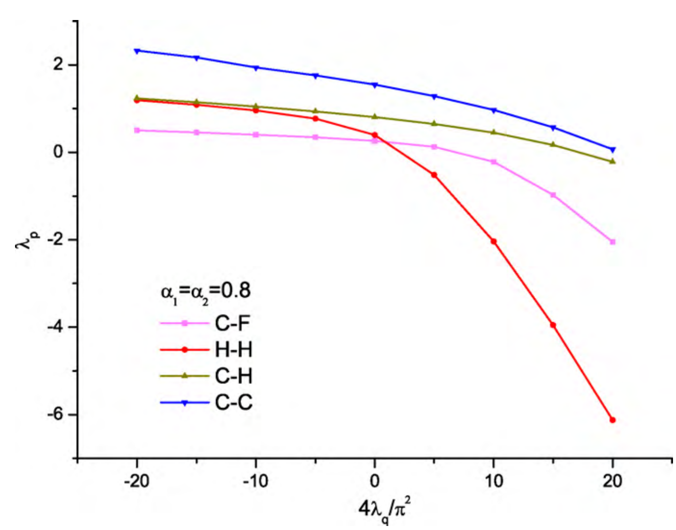

Fig. 2. Critical load λ_p versus $4\lambda_q/\pi^2$ for a non-prismatic bar with linearly varying square cross-section under self-weight ($\alpha_1 = \alpha_2 = 0.8$).

$I(\xi) = (1-0.8\xi)^4, \quad f(\xi) = \int_\xi^1 (1-0.8s)^2\,ds$. For this case, numerical results are displayed in Fig. 2. The trend for a non-prismatic in Fig. 2 is similar to that for a prismatic heavy column (Wang and Ang, 1988).

5. Conclusions

The buckling of prismatic and non-prismatic columns under self-weight and tip force was analyzed. We gave a simple and easy-to-implement semi-analytic approach to solve this problem. That is, with the aid of boundary conditions at both ends, the ordinary differential equation for buckling of elastic columns is transformed into an integral equation, and then to a system of linear equations. The lowest eigenvalue gives the desired buckling load. This provides a convenient tool for optimal design of structural elements. The conclusions obtained in this paper are drawn below:

- The critical buckling load λ_p under a compressive end force decreases with the increment of the taper ratio α_1 or α_2.
- With the increment of the taper ratio α_1 or α_2, the critical buckling load λ_q increases for clamped-free columns under self-weight. For other end supports, λ_q decreases if $\alpha_1 = 0$ or $\alpha_1 = \alpha_2$, but increases if $\alpha_2 = 0$.
- For each end support, the critical buckling load λ_p decreases with the increment of α_q.

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