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# An exact analytical solution for two-dimensional, unsteady, multilayer heat conduction in spherical coordinates

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ABSTRACT

Analytical series solution is proposed for the transient boundary-value problem of *multilayer heat conduction* in $r$–$\theta$ spherical coordinates. Spatially non-uniform, but time-independent, volumetric heat sources may exist in the concentric layers. Proposed solution is valid for any combination of homogenous boundary conditions of the *first* or *second* kind in the $\theta$-direction. However, inhomogeneous boundary conditions of the *first*, *second* or *third* kind may be applied at the inner and outer radial boundaries of the concentric layers. It is noted that the proposed solution is "free" from imaginary eigenvalues. Real eigenvalues are obtained by virtue of precluded explicit dependence of radial eigenvalues on those in the $\theta$-direction. Solution is shown to be relatively simple for the most common spherical geometries–(multilayer) hemisphere and full sphere. An illustrative problem of heat conduction in a three-layer hemisphere is solved. Results along with the isotherms are shown graphically and discussed.



## 1. Introduction

Construction and development using multilayer components is of significant interest due to the added advantage of combining different thermo-physical properties. These components find a wide range of applications in various automotive, space, chemical, civil and nuclear industries. Temperature distribution in such components, with the presence of sinks/sources or convective boundary conditions, may either be obtained analytically or numerically. Though not always available, exact analytical solutions are desirable since: (1) better insight can be gained through the mathematical form of an analytical solution compared to a discrete numerical solution; and (2) these analytical solutions can be used as benchmark or reference results to test and "verify" numerical algorithms and codes. Though useful, analytical solutions to the problem of *multidimensional transient heat conduction* in several multilayer geometries are still lacking. Such solutions are only available for the parallel slabs [1–18] and are, very recently, developed for cylindrical geometries composed of layers [19–23]. Here, we report the first analytical solution to two-dimensional ($r$–$\theta$ spherical coordinates), time-dependent heat conduction problem in concentric layers.

Earlier, Salt [6,7] applied the *orthogonal expansion technique* to solve for unsteady heat conduction in a two-dimensional (2D) Cartesian slab. Mikhailov and Ozisik [8] solved the same problem in a 3D slab. Haji-Sheikh and Beck [12] used the *Green's function approach* to determine the temperature distribution in a 3D two-layer orthotropic slab. Lu et al. [16,19,20] combined the *Laplace transform* and the *Separation of variables* methods to determine the temperature distribution in 2D rectangular and cylindrical media. Analytical solutions in the cylindrical coordinate system were restricted to the $r$–$z$ coordinates [19,20]. Only recently, series solutions based on the *Separation of variables method* in polar coordinates are reported for both pie-slice [21] and annular [22] geometries. These solutions are valid for any combinations of the time-independent, inhomogeneous boundary conditions at the inner and outer radii of the domain. For time-dependent boundary conditions in a multilayer annulus, a solution scheme based on the *Finite Integral Transform method* is recently proposed [23]. In spherical coordinates, analytical solutions to the multilayer heat conduction problem have been limited to 1D, solid sphere with concentric layers [24,25]. Solution methodology in [24] is an extension of the combined *Laplace transform* and the *Separation of variables* method used in Refs. [19,20] and is only applicable to a full sphere. Recently, a series solution is obtained for heat conduction in a three-layer solid aerosol particle heated by laser radiation [25].

Multi-dimensional effects in spherical geometry may result from initial conditions, boundary conditions, heterogeneity in the $\theta$ and/or $\phi$-direction, or due to $\theta$ and/or $\phi$-dependent heat generation terms – $\theta$ or $\phi$-dependent boundary conditions or heat generation terms being the more likely scenario to be encountered in

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**Nomenclature**

| | |
|---|---|
| $a_{imp}$, $b_{imp}$ | coefficients of spherical Bessel functions in radial eigenfunction (see Eq. (31)) |
| $A_{in}$, $B_{in}$, $C_{in}$ | coefficients in Eq. (4) |
| $A_{out}$, $B_{out}$, $C_{out}$ | coefficients in Eq. (5) |
| $C_{1in}$, $C_{2in}$, $C_{1out}$, $C_{2out}$ | coefficients in Eq. (35) dependent on inner and outer surface boundary conditions |
| $D_{mp}$ | coefficient in general solution (Eq. (27)) dependent on initial condition |
| $f_i$ | initial temperature distribution in the $i$th layer at $t = 0$ |
| $f_p(r)$ | particular integral in Eq. (42) |
| $g_i$ | volumetric heat source distribution in the $i$th layer |
| $h_{out}$ | outer surface heat transfer coefficient |
| $J_{m+0.5}$ | Bessel function of the first kind of order $(m + 0.5)$ |
| $k_i$ | thermal conductivity of the $i$th layer |
| $m$ | index |
| $M$ | number of $\theta$-direction eigenfunctions used in the solution |
| $M_{rmp}$ | norms for $r$-direction |
| $P$ | number of radial eigenfunction used in the solution corresponding to each $\theta$-direction eigenvalue |
| $P_m(\mu)$ | Legendre functions of the first kind of degree $m$ |
| $Q_m(\mu)$ | Legendre functions of the second kind of degree $m$ |
| $r$ | radial coordinate |
| $r_i$ | outer radius of the $i$th layer |
| $R_{imp}(\lambda_{imp}r)$ | radial eigenfunctions for the $i$th layer |
| $S$ | volumetric heat source strength in layer-2 of illustrative problem |
| $t$ | time |
| $T_i(r, \theta, t)$ | temperature distribution in the $i$th layer |
| $Y_{m+0.5}$ | Bessel function of the second kind of order $(m + 0.5)$ |
| $x_{ij}$, $y_{ij}$, $j = 1, 2, 3, 4$ | elements of $(2n \times 2n)$ matrix in Eq. (35) |

*Greek symbols*

| | |
|---|---|
| $\alpha_i$ | thermal diffusivity of the $i$th layer |
| $\Delta\lambda$ | window size for evaluation of radial eigenvalues |
| $\theta$ | polar coordinate |
| $\mu$ | defined as $\mu = \cos\theta$ in Eq. (2) |
| $\phi$ | azimuth variable |
| $\beta_m$ | $\theta$-direction eigenvalues (degree of Legendre functions) |
| $\Theta_m(\mu)$ | eigenfunctions in the polar direction |
| $\lambda_{imp}$ | radial eigenvalues |
| $\omega_{1m}$, $\omega_{2m}$ | coefficients in $\Theta_m(\mu)$ Eq. (28) |

*Subscripts and superscripts*

| | |
|---|---|
| $i$ | layer or interface number |
| $ss$ | steady-state |
| $'$ | Differentiation |

practical engineering problems. Time-dependent temperature distribution in a layered spherical fuel element in a pebble bed reactor with $\theta$-dependent convective boundary condition may, for example, benefit from such solutions. Another relevant scenario that may also lead to $\theta$-dependent temperature distribution is that of heat conduction in a *part*-spherical geometry i.e. $0 \le \theta \le \Psi$ and $\Psi < \pi$. Temperature in such a geometry is inherently dependent on $\theta$ due to the geometry itself and therefore, the temperature distribution is a function of at least two variables, $r$ and $\theta$.

This paper presents an analytical series solution for transient boundary-value problem of heat conduction in $r$–$\theta$ spherical coordinates. Proposed solution is applicable in spherical or *part*-spherical multilayer geometries in which temperature does not depend upon the $\phi$ variable, such as (see Fig. 1): spherical cone (a), hemisphere (b), spherical wedge (c), or full sphere (d). Spatially non-uniform (only $r$ and $\theta$-dependent), time-independent volumetric heat sources may be present in the layers. Inhomogeneous, time-independent, $\theta$-dependent boundary conditions of the *first*, *second* or *third* kind may be applied on the inner and outer radial boundaries. However, only homogeneous boundary conditions of the *first* or *second* kind may be applied on the $\theta$-direction boundary surfaces.

## 2. Mathematical formulation

Consider $n$ concentric spherical shells in perfect thermal contact ($r_0 \le r \le r_n$, $0 \le \theta \le \Psi$ and $0 \le \phi \le 2\pi$), as shown schematically in Fig. 2. Different shapes shown in Fig. 1 can be obtained by varying the angle $\Psi$ ($0 < \Psi \le \pi$), such as a spherical cone ($\Psi < \pi/2$), a hemisphere ($\Psi = \pi/2$), a spherical wedge ($\Psi > \pi/2$) and a full sphere ($\Psi = \pi$). All the layers are assumed to be isotropic. Let $k_i$ and $\alpha_i$ be the temperature independent thermal conductivity and thermal diffusivity of the $i$th layer. Initially, at $t = 0$, the $i$th layer is at a specified temperature $f_i(r, \theta)$. For $t > 0$, homogenous boundary conditions of either *first* or *second* kind can be applied on the $\theta = \Psi$ surface. All the three kinds of boundary conditions can be applied on the inner ($i = 1$, $r = r_0$) and the outer ($i = n$, $r = r_n$) radial surfaces. In addition, spatially non-uniform, time-independent heat sources $g_i(r, \theta)$ are switched on in each layer at $t = 0$. Here, none of the initial or boundary conditions or the heat generation rates depend on the azimuth ($\phi$) variable. Hence, the temperature distribution is assumed to be azimuthally symmetric i.e. independent of $\phi$.

Under these assumptions, governing heat conduction equation for the temperature in the $i$th layer $T_i(r, \theta, t)$, along with the boundary and initial conditions, can be written as follows:

Governing equation:

$$\frac{1}{\alpha_i}\frac{\partial T_i}{\partial t}(r,\theta,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_i}{\partial r}(r,\theta,t)\right) + \frac{1}{r^2\sin\theta} \times \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T_i}{\partial\theta}(r,\theta,t)\right) + \frac{g_i(r,\theta)}{k_i}, \quad r_{i-1} \le r \le r_i,\ 1 \le i \le n \tag{1}$$

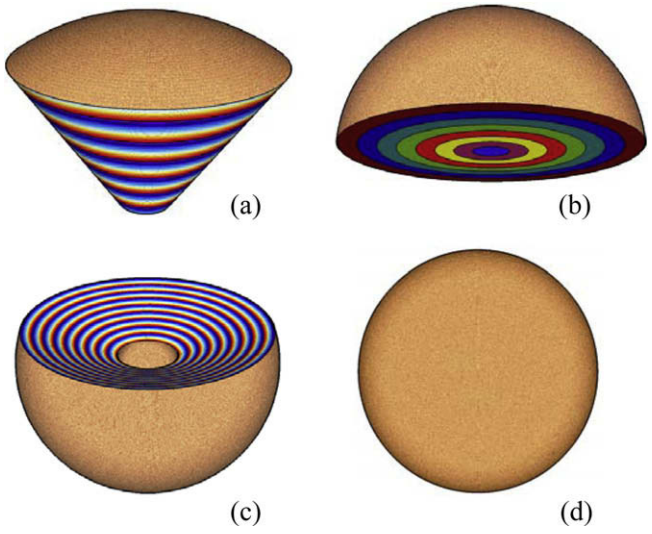



**Fig. 1.** 3D representation of azimuthally symmetric spherical and *part*-spherical multilayer geometries: (a) spherical cone; (b) hemisphere; (c) spherical wedge; and (d) full sphere.

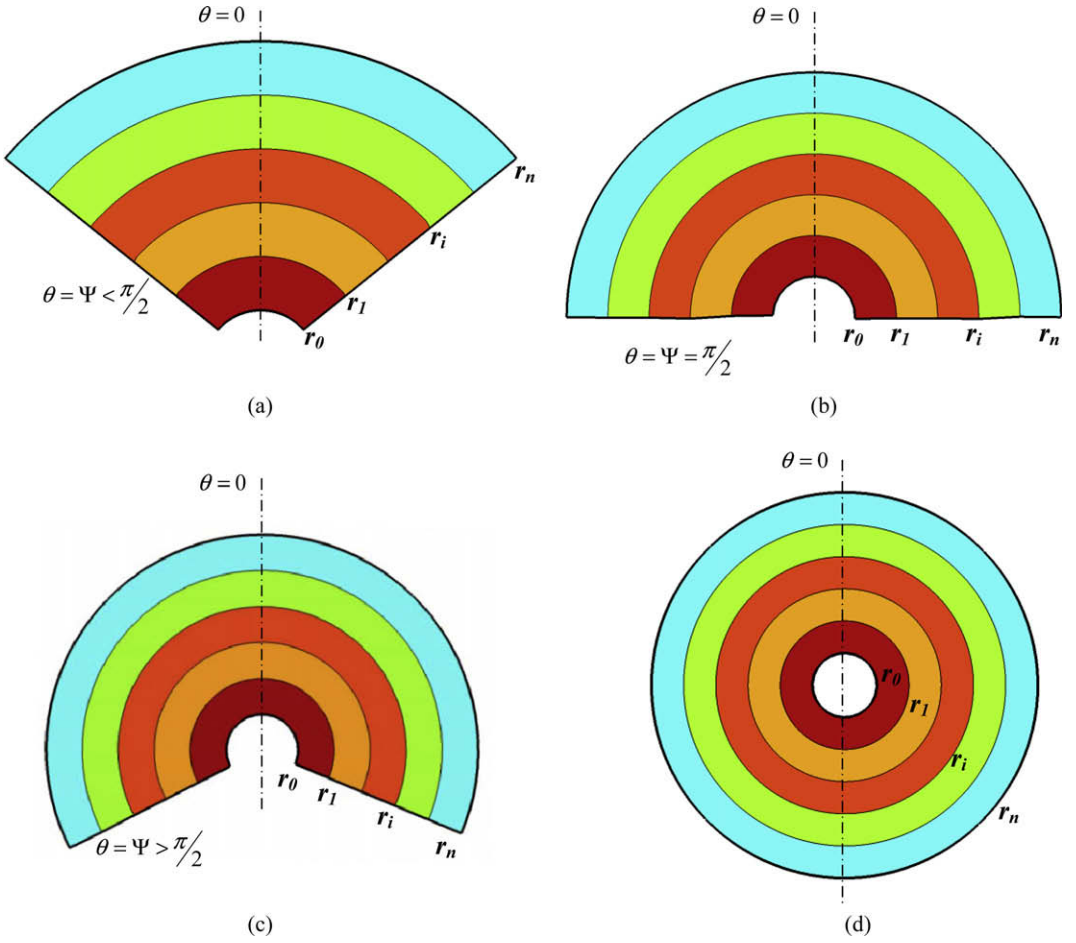



**Fig. 2.** 2D cross-sectional views of $n$ concentric spherical layers ($r_0 \leqslant r \leqslant r_n$ and $0 \leqslant \theta \leqslant \Psi$). Layers are azimuthally symmetric ($0 \leqslant \phi \leqslant 2\pi$) and in perfect thermal contact. Different shapes can be characterized by varying the angle $\Psi$ ($0 < \Psi \leqslant \pi$). (a) spherical cone, $\Psi < \pi/2$; (b) hemisphere, $\Psi = \pi/2$; (c) spherical wedge, $\Psi > \pi/2$ and (d) full sphere, $\Psi = \pi$.

Eq. (1) may be transformed in a more convenient form by defining a new independent variable $\mu$ as

$$\mu = \cos\theta \tag{2}$$

to yield

$$\frac{1}{\alpha_i}\frac{\partial T_i}{\partial t}(r,\mu,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_i}{\partial r}(r,\mu,t)\right) + \frac{1}{r^2} \times \frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial T_i}{\partial \mu}(r,\mu,t)\right) + \frac{g_i(r,\mu)}{k_i}, \quad r_{i-1} \leq r \leq r_i,\ 1 \leq i \leq n \tag{3}$$

Boundary conditions:

- Inner surface of 1st layer ($i = 1$)

$$A_{in}\frac{\partial T_1}{\partial r}(r_0,\mu,t) + B_{in}T_1(r_0,\mu,t) = C_{in}(\mu) \tag{4}$$

- Outer surface of $n$th layer ($i = n$)

$$A_{out}\frac{\partial T_n}{\partial r}(r_n,\mu,t) + B_{out}T_n(r_n,\mu,t) = C_{out}(\mu) \tag{5}$$

- $\theta = \Psi$ surface ($i = 1, 2, \ldots, n$)

$$T_i(r,\mu=\mu_\psi,t) = 0 \text{ or } \frac{\partial T_i}{\partial \mu}(r,\mu=\mu_\psi,t) = 0, \quad \mu_\psi = \cos\Psi,\ \Psi < \pi \tag{6}$$

- Interface of the $i$th layer ($i = 2, \ldots, n$)

$$T_i(r_{i-1},\mu,t) = T_{i-1}(r_{i-1},\mu,t) \tag{7}$$

$$k_i\frac{\partial T_i}{\partial r}(r_{i-1},\mu,t) = k_{i-1}\frac{\partial T_{i-1}}{\partial r}(r_{i-1},\mu,t) \tag{8}$$

- Initial condition:

$$T_i(r,\mu,t=0) = f_i(r,\mu) \tag{9}$$

It is to be noted that boundary conditions either of the *first*, *second* or *third* kind can be imposed on the inner ($r = r_0$) and outer ($r = r_n$) surfaces by choosing the coefficients in Eqs. (4) and (5) appropriately. Furthermore, heat conduction in a geometry with zero inner radius ($r_0 = 0$) can also be solved by assigning zero values to $B_{in}$ and $C_{in}(\mu)$ in Eq. (4).

## 3. Solution methodology

To apply the *separation of variables method*, which is only applicable to homogenous transient problems, the non-homogenous problem described in Section 2 is split into a homogenous transient problem for $\overline{T}_i(r,\mu,t)$ and a non-homogenous steady state problem for $T_{ss,i}(r,\mu)$, where $\overline{T}_i(r,\mu,t) \equiv T_i(r,\mu,t) - T_{ss,i}(r,\mu)$. The governing equations for the two sub-problems are as follows:

### 3.1. Homogeneous transient problem for $\overline{T}_i(r,\mu,t)$

Governing equation:

$$\frac{1}{\alpha_i}\frac{\partial \overline{T}_i}{\partial t}(r,\mu,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \overline{T}_i}{\partial r}(r,\mu,t)\right) + \frac{1}{r^2}\frac{\partial}{\partial \mu} \times \left((1-\mu^2)\frac{\partial \overline{T}_i}{\partial \mu}(r,\mu,t)\right), \quad r_{i-1} \le r \le r_i,\ \mu_\Psi \le \mu \le 1,\ 1 \le i \le n \tag{10}$$

Boundary conditions:

- Inner surface of the 1st layer ($i = 1$)

$$A_{in}\frac{\partial \overline{T}_1}{\partial r}(r_0,\mu,t) + B_{in}\overline{T}_1(r_0,\mu,t) = 0 \tag{11}$$

- Outer surface of the $n$th layer ($i = n$)

$$A_{out}\frac{\partial \overline{T}_n}{\partial r}(r_n,\mu,t) + B_{out}\overline{T}_n(r_n,\mu,t) = 0 \tag{12}$$

- $\theta = \Psi$ surface ($i = 1,2,\ldots,n$)

$$\overline{T}_i(r,\mu=\mu_\Psi,t) = 0 \text{ or } \frac{\partial \overline{T}_i}{\partial \mu}(r,\mu=\mu_\Psi,t) = 0, \quad \mu_\Psi = \cos\Psi,\ \Psi < \pi \tag{13}$$

- Interface of the $i$th layer ($i = 2,\ldots,n$)

$$\overline{T}_i(r_{i-1},\mu,t) = \overline{T}_{i-1}(r_{i-1},\mu,t) \tag{14}$$

$$k_i\frac{\partial \overline{T}_i}{\partial r}(r_{i-1},\mu,t) = k_{i-1}\frac{\partial \overline{T}_{i-1}}{\partial r}(r_{i-1},\mu,t) \tag{15}$$

Initial condition:

$$\overline{T}_i(r,\mu,t=0) = f_i(r,\mu) - T_{ss,i}(r,\mu) \tag{16}$$

### 3.2. Inhomogeneous steady state problem for $T_{ss,i}(r,\mu)$

Governing equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_{ss,i}}{\partial r}(r,\mu)\right) + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial T_{ss,i}}{\partial \mu}(r,\mu)\right) + \frac{g_i(r,\mu)}{k_i} = 0, \quad r_{i-1} \le r \le r_i,\ \mu_\Psi \le \mu \le 1,\ 1 \le i \le n \tag{17}$$

Boundary conditions:

- Inner surface of the 1st layer ($i = 1$)

$$A_{in}\frac{\partial T_{ss,1}}{\partial r}(r_0,\mu) + B_{in}T_{ss,1}(r_0,\mu) = C_{in}(\mu) \tag{18}$$

- Outer surface of the $n$th layer ($i = n$)

$$A_{out}\frac{\partial T_{ss,n}}{\partial r}(r_n,\mu) + B_{out}T_{ss,n}(r_n,\mu) = C_{out}(\mu) \tag{19}$$

- $\theta = \Psi$ surface ($i = 1,2,\ldots,n$)

$$T_{ss,i}(r,\mu=\mu_\Psi) = 0 \text{ or } \frac{\partial T_{ss,i}}{\partial \mu}(r,\mu=\mu_\Psi) = 0, \quad \mu_\Psi = \cos\Psi,\ \Psi < \pi \tag{20}$$

- Interface of the $i$th layer ($i = 2,\ldots,n$)

$$T_{ss,i}(r_{i-1},\mu) = T_{ss,i-1}(r_{i-1},\mu) \tag{21}$$

$$k_i\frac{\partial T_{ss,i}}{\partial r}(r_{i-1},\mu) = k_{i-1}\frac{\partial T_{ss,i-1}}{\partial r}(r_{i-1},\mu) \tag{22}$$

## 4. Solution to the homogenous transient problem

### 4.1. Separation of variables

Substituting the product form for temperature $\overline{T}_i(r,\mu,t)$,

$$\overline{T}_i(r,\mu,t) = R_i(r)\Theta(\mu)\Gamma_i(t) \tag{23}$$

in Eq. (10), and then applying the *separation of variables* method, yield the following ordinary differential equations (ODEs):

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}(\mu)\right] + \beta_m(\beta_m+1)\Theta(\mu) = 0 \tag{24}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_i}{dr}(r)\right) + \left[\lambda_i^2 - \frac{(\beta_m+0.5)^2}{r^2}\right]R_i(r) = 0 \tag{25}$$

$$\frac{1}{\alpha_i}\frac{d\Gamma_i(t)}{dt} + \lambda_i^2\Gamma_i(t) = 0 \tag{26}$$

where $\lambda_i^2$ and $\beta_m$ ($\beta_m + 1$) are the constants of separation. Eq. (24) is the *Legendre's differential equation*, which has linearly independent solutions $P_{\beta_m}(\mu)$ and $Q_{\beta_m}(\mu)$ – *Legendre functions of degree* $\beta_m$, of the *first* and *second* kinds, respectively [26]. Eq. (25) is the *spherical Bessel differential equation*, which has solutions $R_J(\lambda_i r) \equiv \frac{1}{\sqrt{r}}J_{\beta_m+0.5}(\lambda_i r)$ and $R_Y(\lambda_i r) \equiv \frac{1}{\sqrt{r}}Y_{\beta_m+0.5}(\lambda_i r)$ – known as *spherical Bessel functions*, of the *first* and *second* kinds, respectively.

### 4.2. General solution

In view of Eqs. (24)–(26), a general solution for Eq. (10) may be written as:

$$\overline{T}_i(r,\mu,t) = \sum_{m=0}^{\infty}\sum_{p=1}^{\infty} D_{mp}e^{-\alpha_i\lambda_{imp}^2 t}R_{imp}(\lambda_{imp}r)\Theta_m(\mu). \tag{27}$$

In the above equation, $\Theta_m(\mu)$ are eigenfunctions corresponding to the eigenvalue problem in the $\theta$-direction (Eq. (24)), and are given by:

$$\Theta_m(\mu) = \omega_{1m}P_{\beta_m}(\mu) + \omega_{2m}Q_{\beta_m}(\mu) \tag{28}$$

where $\omega_{1m}$ and $\omega_{2m}$ are constants, and

$$\mu_\Psi \le \mu \le 1 \tag{29}$$

For temperature to remain finite, $\omega_{2m}$ must be zero since for all values of $\beta_m$, the $Q_{\beta_m}(\mu)$ functions tend to infinity as $\mu$ approaches 1 [26].

Orthogonality condition for the remaining $\theta$ -direction eigenfunction, $P_{\beta_m}(\mu)$, is:

$$\int_{\mu_\Psi}^{1} P_{\beta_m}(\mu)P_{\beta_l}(\mu)d\mu = \begin{cases} 0 & \text{if } m \ne l \\ N_m & \text{if } m = l \end{cases} \tag{30}$$

Similarly, $R_{imp}(\lambda_{imp}r)$ in the general solution given by Eq. (27) are eigenfunctions corresponding to the eigenvalue problem in the $r$-direction (Eq. (25)), and are given by,

$$R_{imp}(\lambda_{imp}r) = a_{imp}R_J(\lambda_{imp}r) + b_{imp}R_Y(\lambda_{imp}r) \tag{31}$$

Orthogonality condition for the radial eigen functions $R_{imp}(\lambda_{imp}r)$, which is similar to that in Refs. [21–23], is given by,

$$\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_{r_{i-1}}^{r_i} r^2R_{imp}(\lambda_{imp}r)R_{imq}(\lambda_{imq}r)dr = \begin{cases} 0 & \text{if } p \ne q \\ M_{rmp} & \text{if } p = q \end{cases} \tag{32}$$

Moreover, the following relationship between $\lambda_{imp}$ and $\lambda_{1mp}$ must hold for all values of $t$ [6, 11, 21–23] for $\overline{T}_i(r,\mu,t)$ to be continuous at the layer interfaces,

$$\lambda_{imp} = \lambda_{1mp}\sqrt{\alpha_1/\alpha_i}. \tag{33}$$

### 4.3. $\theta$-Direction eigencondition

Only homogenous boundary conditions of the *first* or *second* kind may be applied on $\theta = \Psi$ $(0 < \Psi < \pi)$ surface. Boundary conditions, which are either inhomogeneous or are of the *third* kind, can not be employed in the $\theta$-direction since they produces unconditional mathematical incompatibilities [13]. The $\theta$-direction eigenfunctions $P_{\beta_m}(\mu)$ that satisfy these boundary conditions give the following eigencondition:

$$P_{\beta_m}(\mu = \mu_\Psi) = 0 \text{ or } \frac{\partial P_{\beta_m}}{\partial \mu}(\mu = \mu_\Psi) = 0, \quad -1 < \mu_\psi < 1. \tag{34}$$

Roots of Eq. (34) yield the eigenvalues $\beta_m$ (indexed by an integer $m$).

For (multilayer) hemisphere and for full sphere, which are the most common spherical geometries, the $\theta$-direction eigenvalues $(\beta_m)$ can be determined without solving Eq. (34) by using some of the well known properties of the *Legendre functions*. In the case of a hemisphere $(\Psi = \pi/2, (0 \le \mu \le 1)$, a homogenous boundary condition of the *first kind* at the base (i.e.$P_{\beta_m}(\mu = 0) = 0$) is only satisfied when $\beta_m$ are odd integers, i.e. $\beta_m = 1, 3, 5, \ldots$ Similarly, a homogenous boundary condition of the *second kind* at the base (i.e. $\frac{dP_{\beta_m}}{d\mu}(\mu = 0) = 0$) is only satisfied for even integer values of $\beta_m$ i.e. $\beta_m = 0,2,4\ldots$ [26]. In the case of a full sphere $(\Psi = \pi, -1 \leqslant \mu \leqslant 1)$, eigenfunctions $P_{\beta_m}(\mu)$ at $\mu = -1$ are finite only for non-negative integer values of $\beta_m$, i.e., $\beta_m = 0, 1, 2, 3\ldots$ Fractional values of $\beta_m$ are excluded from the solution due to the non-finiteness of eigenfunctions $P_{\beta_m}(\mu)$ at $\mu = -1$ [26].

### 4.4. Radial eigencondition

Application of the boundary conditions (11) and (12) and the interface conditions (14) and (15) to the radial eigenfunction $R_{imp}(\lambda_{imp} r)$ yields, for each $m$, the following $(2n \times 2n)$ matrix equation:

$$\begin{bmatrix}
C_{1in} & C_{2in} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
x_{11} & x_{12} & x_{13} & x_{14} & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
y_{11} & y_{12} & y_{13} & y_{14} & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & x_{i1} & x_{i2} & x_{i3} & x_{i4} & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & y_{i1} & y_{i2} & y_{i3} & y_{i4} & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & x_{n-1,4} \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & y_{n-1,1} & y_{n-1,2} & y_{n-1,3} & y_{n-1,4} \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & C_{1out} & C_{2out}
\end{bmatrix}
\begin{bmatrix} a_{1mp} \\ b_{1mp} \\ \ldots \\ \ldots \\ a_{imp} \\ b_{imp} \\ \ldots \\ \ldots \\ a_{nmp} \\ b_{nmp} \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ \ldots \\ \ldots \\ 0 \\ 0 \\ \ldots \\ \ldots \\ 0 \\ 0 \end{bmatrix} \tag{35}$$

where

$$\begin{aligned}
C_{1in} &= A_{in} R'_J(\lambda_{1mp} r_0) + B_{in} R_J(\lambda_{1mp} r_0) \\
C_{2in} &= A_{in} R'_Y(\lambda_{1mp} r_0) + B_{in} R_Y(\lambda_{1mp} r_0) \\
x_{i1} &= R_J(\lambda_{imp} r_i) \\
x_{i2} &= R_Y(\lambda_{imp} r_i) \\
x_{i3} &= -R_J(\lambda_{i+1,mp} r_i) \\
x_{i4} &= -R_Y(\lambda_{i+1,mp} r_i) \\
y_{i1} &= k_i R'_J(\lambda_{imp} r_i) \\
y_{i2} &= k_i R'_Y(\lambda_{imp} r_i) \\
y_{i3} &= -k_{i+1} R'_J(\lambda_{i+1,mp} r_i) \\
y_{i4} &= -k_{i+1} R'_Y(\lambda_{i+1,mp} r_i) \\
C_{1out} &= A_{out} R'_J(\lambda_{nmp} r_n) + B_{out} R_J(\lambda_{nmp} r_n) \\
C_{2out} &= A_{out} R'_Y(\lambda_{nmp} r_n) + B_{out} R_Y(\lambda_{nmp} r_n)
\end{aligned}$$

and prime ($'$) denotes differentiation.

In the above matrix equation, all $\lambda_{imp} (i \neq 1)$ may be written in terms of $\lambda_{1mp}$ using Eq. (33). Subsequently, radial eigencondition can be obtained by setting the determinant of the $(2n \times 2n)$ coefficient matrix in Eq. (35) equal to zero. Roots of which, in turn, yield the infinite number of eigenvalues $\lambda_{1mp}$ corresponding to the first layer for each $m$.

In general, transverse eigenvalues for multilayer time-dependent heat conduction problems in Cartesian coordinates may be imaginary. Same is true for 2D $(r, z)$ cylindrical coordinates [21]. These eigenvalues are imaginary due to the explicit dependence of the transverse eigenvalues on those in the remaining direction(s). However, in 2D $r$–$\theta$ spherical coordinate system, dependence of the radial eigenvalues on those in the other direction is not explicit. The same is also true for multilayer heat conduction problems in cylindrical polar coordinate system [21]. Consequently, absence of explicit dependence leads to a complete solution which does not have imaginary radial eigenvalues, and thus $\lambda_{imp}$ are real [22–23].

### 4.5. Determination of coefficients $a_{imp}$ and $b_{imp}$

Coefficients $a_{imp}$ and $b_{imp}$ in the radial eigenfunction $R_{imp}(\lambda_{imp} r)$ (Eq. (31)) are determined from the following recurrence relationship, obtained by imposing the $i$th interface condition (see Eqs. (14) and (15)), valid for $i \in [1, n-1]$,

$$\begin{pmatrix} a_{i+1,mp} \\ b_{i+1,mp} \end{pmatrix} = \begin{pmatrix} R_J(\lambda_{i+1,mp} r_i) & R_Y(\lambda_{i+1,mp} r_i) \\ k_{i+1} R'_J(\lambda_{i+1,mp} r_i) & k_{i+1} R'_Y(\lambda_{i+1,mp} r_i) \end{pmatrix}^{-1} \begin{pmatrix} R_J(\lambda_{imp} r_i) & R_Y(\lambda_{imp} r_i) \\ k_i R'_J(\lambda_{imp} r_i) & k_i R'_Y(\lambda_{imp} r_i) \end{pmatrix} \begin{pmatrix} a_{imp} \\ b_{imp} \end{pmatrix} \tag{36}$$

where $b_{1mp} = -\frac{C_{1in}}{C_{2in}} a_{1mp}$ and $a_{1mp}$ is arbitrary.

Clearly, two sets of eigenfunctions obtained with different $a_{1mp}$ are proportional to each other and are equally valid solutions of the radial eigenvalue problem. Moreover, after the introduction of $D_{mp}$ in the general solution (see Eq. (27)), there is no need to retain $a_{1mp}$ or $\omega_{1m}$ as separate constants. [The above discussion is in fact true for any eigenvalue problem.]

### 4.6. Determination of coefficient $D_{mp}$

Coefficient $D_{mp}$ in Eq. (27) may be obtained by applying the initial condition (16) and then making use of the orthogonality conditions, as follows:

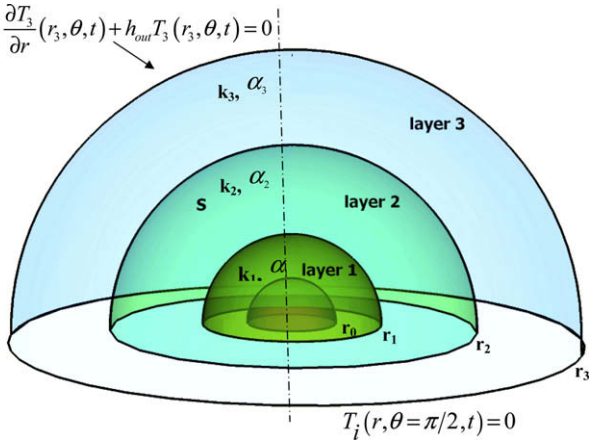



**Fig. 3.** Schematic diagram of the three-layer-hemisphere problem. Mixed boundary condition is applied at the outer radial surface of the hemisphere, while the base and the inner radial surface are kept at zero temperatures. Heat source of strength $S$ is assumed to be uniformly distributed in layer-2.

$$D_{mp} = \frac{1}{M_{rmp}N_m}\sum_{i=1}^{n}\frac{k_i}{\alpha_i}\int_{\mu_\psi}^{1}\int_{r_{i-1}}^{r_i} r^2 R_{imp}(\lambda_{imp}r)P_{\beta_m}(\mu)\overline{T}_i(r,\mu,t = 0)drd\mu \quad (37)$$

## 5. Solution to the inhomogeneous steady state problem

The inhomogeneous steady state problem is solved using eigenfunction expansion method. The steady state temperature distribution $T_{ss,i}(r,\mu)$, governed by Eq. (17), may be written as a generalized Fourier series in terms of $\theta$-direction eigenfunctions $P_{\beta_m}(\mu)$,

$$T_{ss,i}(r,\mu) = \sum_{m=0}^{\infty}\widehat{T}_{im}(r)P_{\beta_m}(\mu) \quad (38)$$

Substituting Eq. (38) in Eq. (17) leads to an ODE for $\widehat{T}_{im}(r)$,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\widehat{T}_{im}}{dr}(r)\right) - \frac{\beta_m(\beta_m+1)}{r^2}\widehat{T}_{im}(r) + \frac{\hat{g}_{im}(r)}{k_i} = 0 \quad (39)$$

Note that the source term $g_i(r,\mu)$ is expanded in a generalized Fourier series as:

$$g_i(r,\mu) = \sum_{m=0}^{\infty}\hat{g}_{im}(r)P_m(\mu) \quad (40)$$

where

$$\hat{g}_{im}(r) = \frac{1}{N_m}\int_{\mu_\psi}^{1} g_i(r,\mu)P_m(\mu)d\mu. \quad (41)$$

Similarly, $C_{in}(\mu)$ and $C_{out}(\mu)$ in boundary conditions given by Eqs. (18) and (19) may be expanded in a generalized Fourier series to yield boundary conditions for ODE given in Eq. (39). Interface conditions for $T_{ss,i}(r,\mu)$, given in Eqs. (21) and (22), are also valid for $\hat{T}_{im}(r)$.

Solution for Eq. (39) may be written as:

$$\widehat{T}_{im}(r) = A_{ss,i}r^{\beta_m} + B_{ss,i}r^{-\beta_m-1} + f_p(r) \quad (42)$$

where $f_p(r)$ is particular integral that can be obtained by application of method of variation of parameters or method of undetermined coefficients. Constants $A_{ss,i}$ and $B_{ss,i}$ may be evaluated using boundary and interface conditions for $\widehat{T}_{im}(r)$.

## 6. Illustrative example

A three-layer hemispherical region ($r_0 \le r \le r_3$, $0 \le \theta \le \pi/2$; see Fig. 3) is initially at a uniform zero temperature. For time $t > 0$, the base of the hemisphere at $\theta = \pi/2$ as well as inner radial surface at $r = r_0$ is maintained at a uniform and constant (zero) temperature, while heat is convected into ambient, also at zero temperature, at the outer radial surface at $r = r_3$. These boundary conditions lead to $A_{in} = 0$, $B_{in} = 1$, $A_{out} = k_3$, $B_{out} = h_{out}$, $C_{in} = 0$ and $C_{out} = 0$. In addition, uniformly distributed heat source of magnitude $S$ is turned on at $t = 0$ in the second (middle) layer.

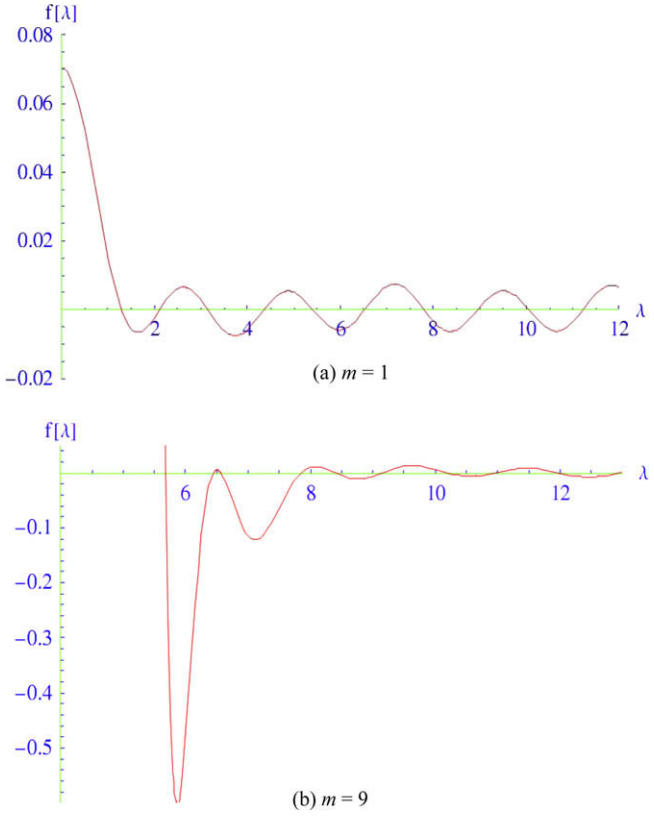



**Fig. 4.** Plot of eigencondition $f[\lambda]$ against $\lambda$ for $m = 1$ and 9. Roots of the transcendental eigencondition are obtained by using Mathematica [27] and are graphically verified by plotting $f[\lambda]$.

Parameter values used for this problem are,

$$k_2/k_1 = 2, k_3/k_1 = 4; \alpha_2/\alpha_1 = 4, \alpha_3/\alpha_1 = 9; r_1/r_0 = 2,$$
$$r_2/r_0 = 4, r_3/r_0 = 6, \text{Biot number } Bi_{out} \equiv h_{out}r_0/k_1 = 1.$$

It should be noted that, in the results that follow, $r$, $t$, and $T(r,\theta,t)$ are in the units of $r_0$, $r_0^2/\alpha_1$ and $Sr_0^2/k_1$, respectively.

Steady-state solution for this problem is,

$$T_{ss,i}(r,\mu) = \sum_{m=1,3,5\ldots}^{\infty}\widehat{T}_{im}(r)P_m(\mu), \quad i = 1,2,3 \quad (43)$$

where

$$\widehat{T}_{im}(r) = A_{ss,i}r^m + B_{ss,i}r^{-m-1}, \quad i = 1,3 \quad (44)$$
$$\widehat{T}_{2m}(r) = A_{ss,1}r^m + B_{ss,1}r^{-m-1} + c_s r^2, \quad i = 2 \quad (45)$$

and $c_s = \frac{-(2m+1)S}{k_2(6-m(m+1))}\int_0^1 P_m(\mu)d\mu$.

**Table 1**
Radial eigenvalues $\lambda_{1mp}$ for the three-layer illustrative problem.

| $p$ | $m = 1$ | $m = 3$ | $m = 5$ | $m = 7$ | $m = 9$ | $m = 11$ | $m = 13$ | $m = 15$ | $m = 17$ | $m = 19$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.29277 | 2.34254 | 3.46003 | 4.57605 | 5.6782 | 6.76164 | 7.8274 | 8.87976 | 9.92315 | 10.9608 |
| 2 | 2.10969 | 2.99396 | 4.12218 | 5.29595 | 6.44482 | 7.54839 | 8.63877 | 9.73342 | 10.8299 | 11.9244 |
| 3 | 3.14826 | 3.73857 | 4.5793 | 5.53735 | 6.56868 | 7.66152 | 8.77274 | 9.87789 | 10.9744 | 12.0634 |
| 4 | 4.36281 | 4.88525 | 5.7218 | 6.74811 | 7.85175 | 8.97043 | 10.0924 | 11.2195 | 12.3513 | 13.4847 |
| 5 | 5.41541 | 5.81587 | 6.48474 | 7.36378 | 8.40086 | 9.53865 | 10.7168 | 11.8973 | 13.0654 | 14.2112 |
| 6 | 6.53354 | 6.88794 | 7.48193 | 8.25761 | 9.15505 | 10.1229 | 11.1270 | 12.1508 | 13.1895 | 14.2495 |
| 7 | 7.79238 | 8.09242 | 8.61171 | 9.32296 | 10.1972 | 11.2013 | 12.2902 | 13.4098 | 14.5239 | 15.6318 |
| 8 | 8.95792 | 9.20332 | 9.63097 | 10.2211 | 10.9523 | 11.8052 | 12.7653 | 13.8235 | 14.9595 | 16.1341 |
| 9 | 10.0519 | 10.2867 | 10.6986 | 11.2713 | 11.9841 | 12.8135 | 13.7329 | 14.7139 | 15.7297 | 16.7616 |
| 10 | 11.2240 | 11.4327 | 11.8008 | 12.3173 | 12.9693 | 13.7440 | 14.6306 | 15.6181 | 16.6902 | 17.8198 |

The constants $A_{ss,i}$ and $B_{ss,i}$ ($i = 1, 2$ and $3$) in Eqs. (44) and (45) can be evaluated by applying the steady-state interface and boundary conditions, which result in the following matrix equation:

$$\begin{bmatrix} A_{ss,1} \\ B_{ss,1} \\ A_{ss,2} \\ B_{ss,2} \\ A_{ss,3} \\ B_{ss,3} \end{bmatrix} = \begin{bmatrix} (A_{in}m + B_{in}r_0)r_0^{m-1} & \{-A_{in}(m+1) + B_{in}r_0\}r_0^{-m-2} & 0 & 0 & 0 & 0 \\ r_1^m & r_1^{-m-1} & -r_1^m & -r_1^{-m-1} & 0 & 0 \\ 0 & 0 & r_2^m & r_2^{-m-1} & -r_2^m & -r_2^{-m-1} \\ k_1 m r_1^{m-1} & -k_1(m+1)r_1^{-m-2} & -k_2 m r_1^{m-1} & k_2(m+1)r_1^{-m-2} & 0 & 0 \\ 0 & 0 & k_2 m r_2^{m-1} & -k_2(m+1)r_2^{-m-2} & -k_3 m r_2^{m-1} & k_3(m+1)r_2^{-m-2} \\ 0 & 0 & 0 & 0 & (A_{out}m + B_{out}r_3)r_3^{m-1} & \{-A_{out}(m+1) + B_{out}r_3\}r_3^{-m-2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ c_s r_1^2 \\ -c_s r_2^2 \\ 2k_2 c_s r_1 \\ -2k_2 c_s r_2 \\ 0 \end{bmatrix} \tag{46}$$

There exist infinite number of radial eigenvalues (indexed by $p$) for the first layer, $\lambda_{1mp}$, for each $m$. These eigenvalues ($\lambda_{1mp}$) are calculated by solving the transcendental eigencondition, corresponding to Eq. (35) i.e. $f[\lambda] = 0$, with the help of Mathematica [27], a commercial mathematical package. Resulting eigenvalues for various values of $m$ and $p$ are shown in Table 1. Roots are searched in a user-defined window of size $\Delta\lambda$ using in-built functions. Successive eigenvalues are obtained by marching forward in steps of $\Delta\lambda$ starting from zero. Since the roots are not distributed uniformly, the window size has to be kept very small in order not to miss any eigenvalue. Moreover, resulting eigenvalues are verified graphically to make sure that all eigenvalues within the interval were indeed captured. In Fig. 4, the variation of eigencondition $f[\lambda]$ with $\lambda$ is shown graphically for $m = 1$ and $m = 9$ in order to demonstrate the viability of root-finding mechanism employed. The abovementioned scheme is not very efficient because a very small window size is required. Several methods have been reported in literature to efficiently compute eigenvalues for 2D Cartesian multilayer problems [12,14]. Further research is necessary to develop an efficient and automated scheme for the current problem, which also guarantees that all eigenvalues are captured.

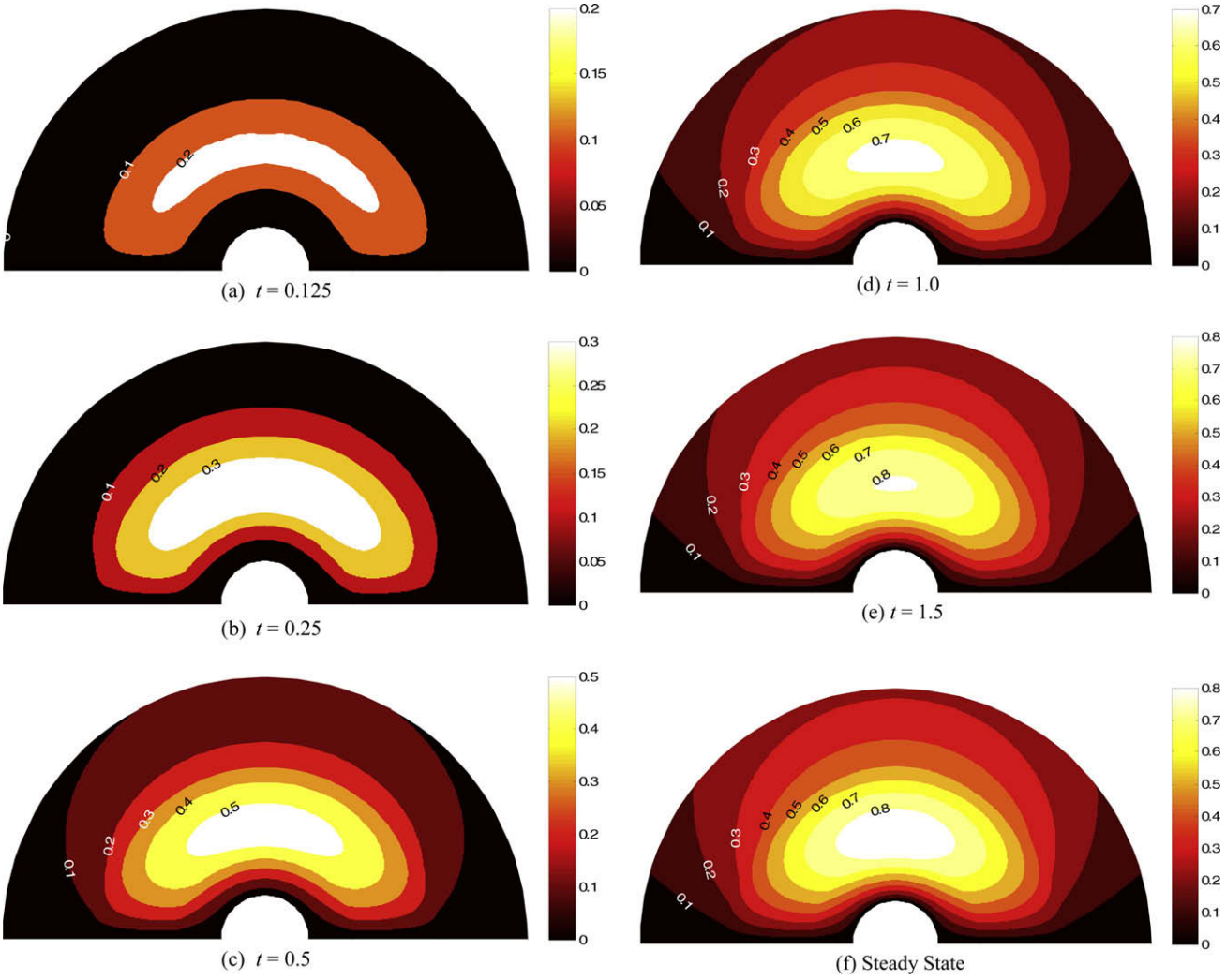



**Fig. 5.** Isotherms in the three-layer hemisphere at different times. Two-dimensional contours are plotted for (any) cross-sectional plane which cuts through the axis of the hemisphere.

## 7. Results

The transient part of the series solution for the illustrative problem described in Section 6 is truncated at $p = P$ and $m = M$, leading to,

$$\overline{T}_i(r,\mu,t) = \sum_{m=1,3,5\ldots}^{OddM}\sum_{p=1}^{P} D_{mp} e^{-\alpha_1 \lambda_{1mp}^2 t}[a_{imp}R_J(\lambda_{imp}r) + b_{imp}R_Y(\lambda_{imp}r)]P_m(\mu) + \varepsilon_i(r,\mu,t,M,P) \quad (47)$$

where $\varepsilon_i(r,\mu,t,M,P)$ is the truncation error.

Since $\lambda_{1mp}$ increases with increasing $m$ and $p$, it is obvious that for a given $M$ and $P$, maximum truncation error occurs at $t = 0$. Moreover, since $\overline{T}_i(r,\mu,t=0) = -T_{ss,i}(r,\mu)$, therefore,

$$\varepsilon_i(r,\mu,t=0,M,P) = T_{ss,i}(r,\mu) + \sum_{m=1,3,5\ldots}^{OddM}\sum_{p=1}^{P} D_{mp}[a_{imp}R_J(\lambda_{imp}r) + b_{imp}R_Y(\lambda_{imp}r)]P_m(\mu) \quad (48)$$

Numerical results show that error progressively decreases with increasing values of $M$ and $P$. Moreover, for $M = 19$ and $P = 10$, trun-

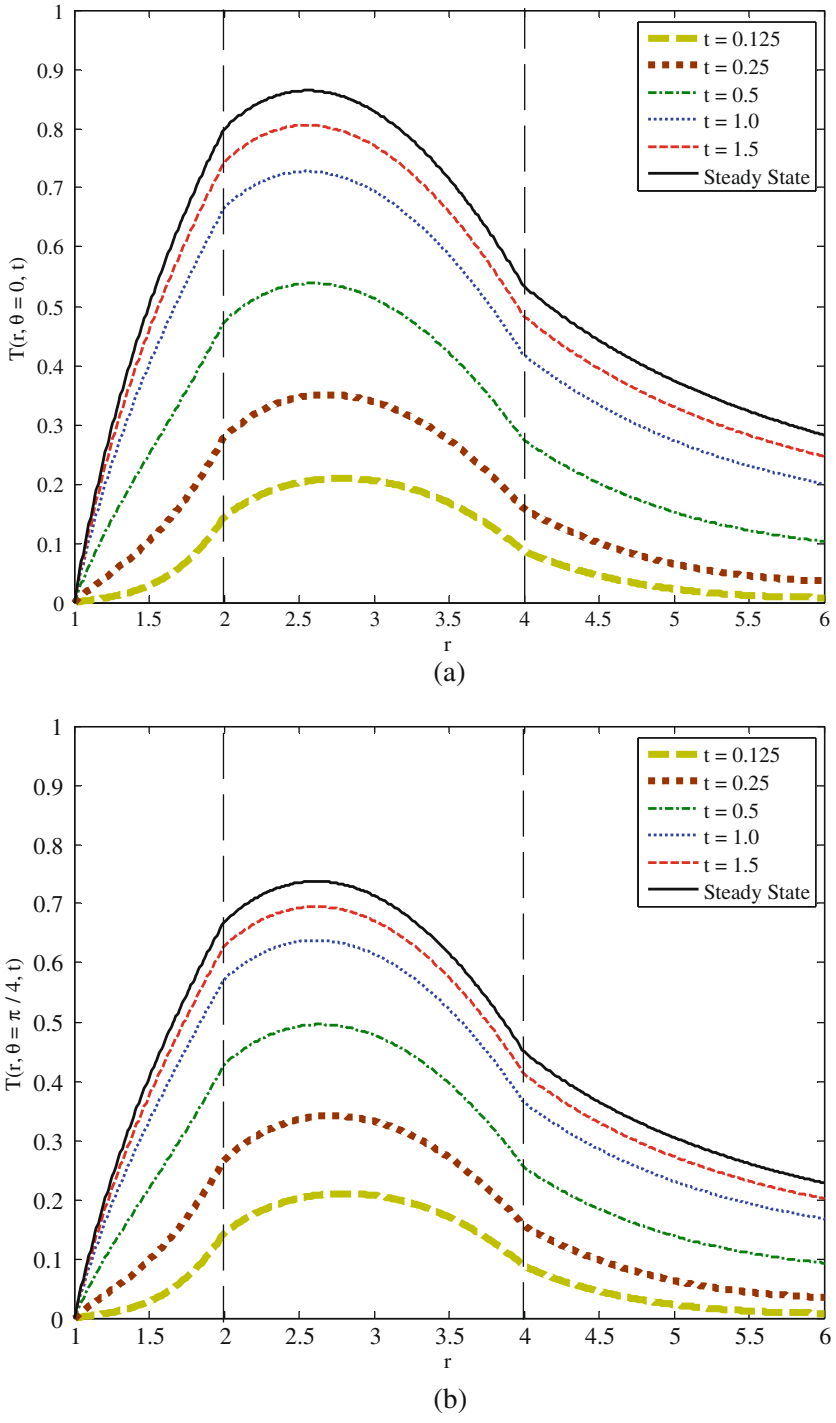
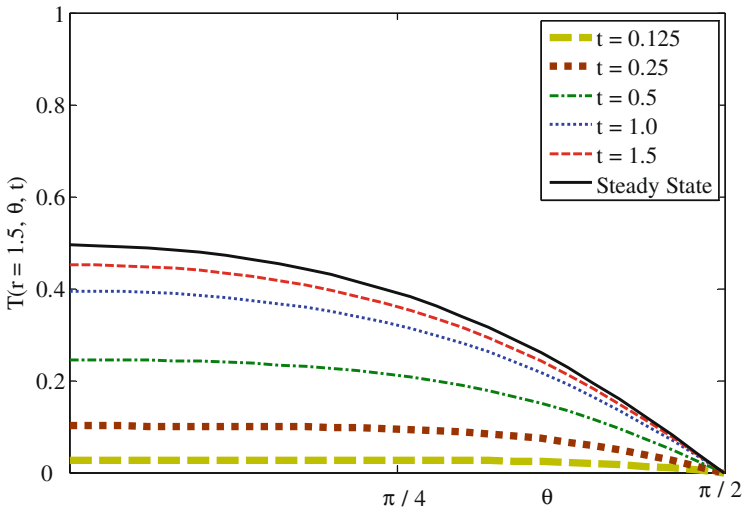
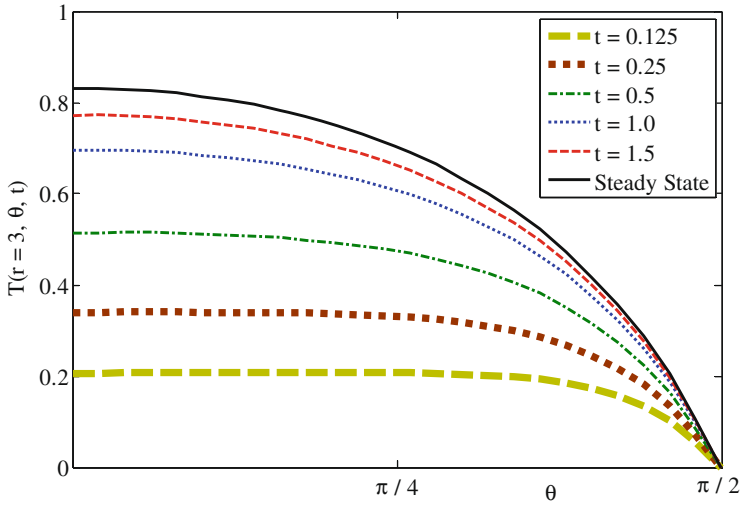


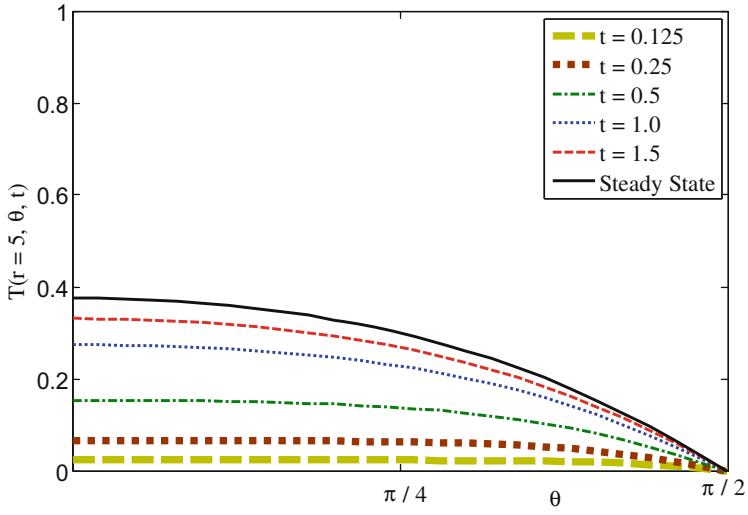

**Fig. 6.** Transient temperature distribution in the radial direction along $\theta = 0$ and $\theta = \pi/4$.




(a)




(b)




(c)

**Fig. 7.** Transient temperature distributions in polar direction at the mid-sections of the three layers in the hemisphere: (a) $r = 1.5$, (b) $r = 3.0$, and (c) $r = 5.0$.

cation error becomes reasonably small (order of $10^{-3}$) and therefore, series is truncated for these values of $M$ and $P$.

Isotherms in the three-layer hemispherical region are shown for different values of $t$ in Fig. 5. Additionally, temperature variations along radial and polar directions are shown in Figs. 6 and 7, respectively. The steady-state solution is also shown for all the cases. It may be noted that the unsteady isotherms (in Fig. 5) and temperature variation in the radial direction (in Fig. 6) show jump in derivative at the layer interfaces due to step change in material properties.

## 8. Conclusions

In this paper, exact analytical solution to the 2D, transient heat conduction problem in spherical coordinate ($r$-$\theta$) with multiple layers in the radial direction is presented. The solution is valid for a variety of spherical and *part*-spherical, azimuthally symmetric, multilayer geometries, such as spherical cone and wedge, hemisphere and full sphere. In fact, the solution gets significantly simpler in more practically relevant spherical geometry – hemisphere and full sphere – because of integer eigenvalues in the $\theta$ direction. Proposed solution is valid for any combination of homogenous boundary condition of the *first* or *second* kind at the $\theta$ -boundaries. However, inhomogeneous boundary condition of the *first*, *second* or the *third* kind can be applied in the radial direction. Proposed solution is also applicable to a geometry with zero inner radius. Numerical evaluation of the double series solution shows that a reasonable number of terms are sufficient to obtain results with acceptable errors for engineering applications.

It is noted that the solution of multilayer, two-dimensional heat conduction problem in spherical coordinates is not analogous to the corresponding problem in multi-dimensional Cartesian coordinates (or 2D cylindrical $r$–$z$ coordinates). In the spherical coordinates, dependence of the radial eigenvalues on those in the polar direction is not explicit. Absence of explicit dependence leads to a complete solution which does not have imaginary radial eigenvalues.

The solution of the multilayer heat conduction problem in the spherical coordinates can not be trivially deduced from the corresponding problems in the cylindrical or polar coordinates. Reason being that except for the special cases of $\Psi = \pi/2$ and $\Psi = \pi$, an eigenvalue problem has to be solved in the angular direction to obtain eigenvalues $\beta_m$. This leads to differences in computations of radial eigenvalues due to their dependence on the angular eigenvalues $\beta_m$.

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