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Numerical solution of fractionally damped beam by homotopy perturbation method

Research Article

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Abstract: This paper investigates the numerical solution of a viscoelastic continuous beam whose damping behaviours are defined in term of fractional derivatives of arbitrary order. The Homotopy Perturbation Method (HPM) is used to obtain the dynamic response. Unit step function response is considered for the analysis. The obtained results are depicted in various plots. From the results obtained it is interesting to note that by increasing the order of the fractional derivative the beam suffers less oscillation. Similar observations have also been made by keeping the order of the fractional derivative constant and varying the damping ratios. Comparisons are made with the analytic solutions obtained by Zu-feng and Xiao-yan [Appl. Math. Mech. 28, 219 (2007)] to show the effectiveness and validation of this method.

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1. Introduction

In recent years, fractional calculus has been used to model physical and engineering problems in fields such as solid mechanics, fluid mechanics, biology, physics, and other areas of engineering and science. Since, it is too difficult to obtain the exact solution of a fractional differential equation so, one may need a reliable and efficient numerical technique for solving fractional differential equations. Many important works on fractional calculus have been reported in the last few decades and several excellent books have been written by different authors representing the scope and various aspects of this field such as: Kiryakov [37], Golmankhaneh [2], Baleanu et al. [6, 7], Miller and Ross [22], Oldham and Spanier [14], Podlubny [8], and Samko et al. [31]. These books also give an extensive review of fractional derivative and fractional differential equations which may help the reader understand the basic concepts of fractional calculus.Many authors have developed various methods to solve ordinary and partial fractional differential equations integral to physical systems. Some commonly used methods are the Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), Differential Transform Method (DTM), etc. which are described in [1, 30, 32–36] and the references mentioned therein.

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Some other related works are reviewed and cited here for a better understanding of the present investigation. Half-order fractional derivative models of viscoelastically damped structures have been excellently studied by Bagley and Torvik [28, 29]. Laplace transform is considered in [28] to find response characteristics. Also, Koeller [26] has used a fractional model to describe creep and relaxation functions for viscoelastic materials. In [15, 16] Fourier transformation is used to analyse the damping description of the impulse response function of oscillators with fractional derivatives. Time domain finite element analysis of viscoelastic structures with fractional derivatives is clearly explained in [20]. The eigenvector expansion method is successfully implemented in [17] to find the solution of dynamic systems containing fractional derivatives. Various numerical methods are applied in [4, 18, 20, 21, 27] to find the responses of a fractionally damped system.

Recently, the homotopy perturbation method has been found to be a powerful tool for analysing this type of system involving fractional derivatives. The Homotopy Perturbation Method (HPM) was first developed by Ji-Huan He in 1999 [9–13] and many authors applied this method to solve various linear and non-linear functional equations of scientific and engineering problems. The solution is considered as the sum of infinite series, which converges rapidly to accurate solutions. In the homotopy technique (in topology), a homotopy is constructed with an embedding parameter which is considered as a "small parameter". Very recently the homotopy perturbation method has been applied to a wide class of physical problems [3, 5, 9, 11, 13, 24, 25].

In this analysis, the homotopy perturbation method is used to handle the dynamic analysis of a fractionally damped viscoelastic continuous beam. The same problem is studied by Zu-feng and Xiao-yan [19] using the adomain decomposition method. A damping factor is defined with a fractional derivative of an arbitrary order. In the following sections, preliminaries are described first, followed by the implementation of HPM for fractionally damped viscoelastic beam. Then the response analysis for a unit step load is presented and finally numerical examples and conclusions are given.

2. Preliminaries

In this section, we present some notations, definitions and preliminary facts which are used further in this paper [8, 14, 22, 31, 37].

Definition 1 (Riemann-Liouville fractional integral).

There are several definitions of fractional integrals. The most commonly used is by Riemann-Liouville and Caputo [8]. The Riemann-Liouville integral operator J^α of order $\alpha \geq 0$, is defined by

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, t > 0.$$

Definition 2 (Caputo derivative).

The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$\begin{aligned} D^\alpha f(t) &= J^{m-\alpha} D^m f(t) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau) d\tau}{(t-\tau)^{\alpha+1-m}}, & m-1 < \alpha < m, m \in N \\ \frac{d^m}{dx^m} f(t), & \alpha = m, m \in N \end{cases} \end{aligned}$$

where, the parameter α is the order of the derivative and is allowed to be real or complex. In this paper, only real and positive α will be considered. For the Caputo's derivative we have

$$D^\alpha C = 0, \; C \text{ is a constatnt}$$

$$D^\alpha t^\beta = \begin{cases} 0, & (\beta \leq \alpha - 1) \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & (\beta > \alpha - 1) \end{cases}$$

Similar to integer-order differentiation, Caputo's fractional differentiation is a linear operation:

$$D^\alpha (\lambda f(t) + \mu g(t)) = \lambda D^\alpha f(t) + \mu D^\alpha g(t),$$

where,λ, μ are constants and satisfies the so called Leibnitz rule:

$$D^\alpha (g(t) f(t)) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} g^{(k)}(t) D^{\alpha-k} f(t),$$

if $f(\tau)$is continuous in $[0, t]$ and $g(\tau)$ has $n+1$continuous derivatives in $[0, t]$.

Definition 3 (Mittage-Leffer function).

A two-parameter function of the Mittage-Leffer type is defined by the series expansion [8]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, (\alpha > 0, \beta > 0).$$

3. Application of HPM [10, 11] to a fractionally damped viscoelastic beam

To develop numerical schemes for a fractionally damped viscoelastic beam [19] let us consider a linear differential equation which describes the dynamics of the system with the damping as an arbitrary fractional derivative of order α

$$\rho A\frac{\partial^2 v}{\partial t^2}+c\frac{\partial^\alpha v}{\partial t^\alpha}+EI\frac{\partial^4 v}{\partial x^4}=F(x,t) \tag{1}$$

where ρ, A, c, E and I represent the mass density, cross sectional area, damping coefficient per unit length, Young's modulus of elasticity and moment of inertia of the beam respectively. $F(x,t)$ is the externally applied force and $v(x,t)$ is the transverse displacement. $\frac{\partial^\alpha}{\partial t^\alpha}$ is the fractional derivative of order $\alpha\in(0,1)$ of the displacement function $v(x,t)$. Initial conditions are considered as $v(x,0)=0$ and $\dot v(x,0)=0$. Homogeneous initial conditions are taken here to compare the solution obtained by the present HPM with the solution of [19].
Equation (1) can be written as

$$\frac{\partial^2 v}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\alpha v}{\partial t^\alpha}+\frac{EI}{\rho A}\frac{\partial^4 v}{\partial x^4}=\frac{F(x,t)}{\rho A}. \tag{2}$$

According to HPM, we may construct a simple homotopy for an embedding parameter $p\in[0,1]$ as follows

$$\begin{aligned}&(1-p)\frac{\partial^2 v}{\partial t^2}+\\ &p\left(\frac{\partial^2 v}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\alpha v}{\partial t^\alpha}+\frac{EI}{\rho A}\frac{\partial^4 v}{\partial x^4}-\frac{F(x,t)}{\rho A}\right)=0,\quad p\in[0,1]\end{aligned} \tag{3}$$

or

$$\frac{\partial^2 v}{\partial t^2}+p\left(\frac{c}{\rho A}\frac{\partial^\alpha v}{\partial t^\alpha}+\frac{EI}{\rho A}\frac{\partial^4 v}{\partial x^4}-\frac{F(x,t)}{\rho A}\right)=0. \tag{4}$$

Here, p is considered as a small homotopy parameter $0\le p\le 1$. For $p=0$, Equations (3) and (4) become a linear equation i.e. $\frac{\partial^2 v}{\partial t^2}=0$, which is easy to solve. For $p=1$, Equations (3) and (4) turn out to be same as the original Equation (1) or (2). This is called deformation in topology. $\frac{\partial^2 v}{\partial t^2}$ and $\frac{c}{\rho A}\frac{\partial^\alpha v}{\partial t^\alpha}+\frac{EI}{\rho A}\frac{\partial^4 v}{\partial x^4}-\frac{F(x,t)}{\rho A}$ are called homotopic.
Next, we can assume the solution of Equation (3) or (4) as a power series expansion in p as

$$v(x,t)=v_0(x,t)+pv_1(x,t)+p^2v_2(x,t)+p^3v_3(x,t)+\cdots, \tag{5}$$

where $v_i(x,t)$ for $i=0,1,2,\ldots$ are functions yet to be determined. Substituting Equation (5) into Equation (3) or (4), and equating the terms with the identical power of p we can obtain a series of equations of the form

$$p^0 := \frac{\partial^2 v_0}{\partial t^2}=0, \tag{6}$$

$$p^1 : \frac{\partial^2 v_1}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\alpha v_0}{\partial t^\alpha}+\frac{EI}{\rho A}\frac{\partial^4 v_0}{\partial x^4}-\frac{F(x,t)}{\rho A}=0, \tag{7}$$

$$p^2 : \frac{\partial^2 v_2}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\alpha v_1}{\partial t^\alpha}+\frac{EI}{\rho A}\frac{\partial^4 v_1}{\partial x^4}=0, \tag{8}$$

$$p^3 : \frac{\partial^2 v_3}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\alpha v_2}{\partial t^\alpha}+\frac{EI}{\rho A}\frac{\partial^4 v_2}{\partial x^4}=0, \tag{9}$$

$$p^4 : \frac{\partial^2 v_4}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\alpha v_3}{\partial t^\alpha}+\frac{EI}{\rho A}\frac{\partial^4 v_3}{\partial x^4}=0, \tag{10}$$

and so on.
Choosing initial approximation $v_0(x,0)=0$ and applying the operator L_{tt}^{-1} (which is the inverse of the operator $L_{tt}=\frac{\partial^2}{\partial t^2}$) on both sides of Equations (6) to (10) one may obtain the following equations

$$v_0(x,t)=0, \tag{11}$$

$$v_1(x,t)=L_{tt}^{-1}\left(-\frac{c}{\rho A}\frac{\partial^\alpha v_0}{\partial t^\alpha}-\frac{EI}{\rho A}\frac{\partial^4 v_0}{\partial x^4}+\frac{F(x,t)}{\rho A}\right), \tag{12}$$

$$v_2(x,t)=L_{tt}^{-1}\left(-\frac{c}{\rho A}\frac{\partial^\alpha v_1}{\partial t^\alpha}-\frac{EI}{\rho A}\frac{\partial^4 v_1}{\partial x^4}\right), \tag{13}$$

$$v_3(x,t)=L_{tt}^{-1}\left(-\frac{c}{\rho A}\frac{\partial^\alpha v_2}{\partial t^\alpha}-\frac{EI}{\rho A}\frac{\partial^4 v_2}{\partial x^4}\right), \tag{14}$$

$$v_4(x,t)=L_{tt}^{-1}\left(-\frac{c}{\rho A}\frac{\partial^\alpha v_3}{\partial t^\alpha}-\frac{EI}{\rho A}\frac{\partial^4 v_3}{\partial x^4}\right), \tag{15}$$

and so on.
Now substituting these terms in Equation (5) with $p\to 1$ one may get the approximate solution of Equation (1) as follows.

$$\begin{aligned}&v(x,t)=v_0(x,t)+v_1(x,t)+v_2(x,t)+v_3(x,t)+\\ &+v_4(x,t)+\cdots\end{aligned} \tag{16}$$

The solution series converge very rapidly. Proof of convergence of the above series may be found in [10, 11]. The rapid convergence means that only a few terms are required to get the approximate solutions.

4. Response analysis

Similar to [19] the external applied force $F(x,t)$ is considered as

$$F(x,t)=f(x)g(t),$$

where $f(x)$ is a specified space dependent deterministic function, and $g(t)$ is a time dependent process. We will now consider the response of the beam to a unit step load of the form $g(t)=Bu(t)$ where $u(t)$ is the Heaviside function and B is a constant. By using HPM we have

$$v_0(x,t)=0, \tag{17}$$

$$v_1(x,t)=\frac{fB}{\rho A}\frac{t^2}{2}, \tag{18}$$

$$v_2(x,t)=-\frac{cfB}{\rho^2A^2}\frac{t^{4-\alpha}}{\Gamma(5-\alpha)}-\frac{EIBf^{(4)}}{\rho^2A^2}\frac{t^4}{\Gamma(5)}, \tag{19}$$

$$\begin{aligned} v_3(x,t)=&\frac{c^2fB}{\rho^3A^3}\frac{t^{6-2\alpha}}{\Gamma(7-2\alpha)}+\frac{2cEIBf^{(4)}}{\rho^3A^3}\frac{t^{6-\alpha}}{\Gamma(7-\alpha)}+\\ &+\frac{E^2I^2Bf^{(8)}}{\rho^3A^3}\frac{t^6}{\Gamma(7)},\end{aligned} \tag{20}$$

$$\begin{aligned} v_4(x,t)=&-\frac{c^3fB}{\rho^4A^4}\frac{t^{8-3\alpha}}{\Gamma(9-3\alpha)}-\frac{3c^2EIBf^{(4)}}{\rho^4A^4}\frac{t^{8-2\alpha}}{\Gamma(9-2\alpha)}-\\ &\frac{3cE^2I^2Bf^{(8)}}{\rho^4A^4}\frac{t^{8-\alpha}}{\Gamma(9-\alpha)}-\frac{E^3I^3Bf^{(12)}}{\rho^4A^4}\frac{t^8}{\Gamma(9)}\end{aligned} \tag{21}$$

and so on, where $f^{(i)}=\frac{\partial^i f}{\partial x^i}$.
In a similar manner the rest of the components can be obtained. Therefore, the solution can be written in its general form as

$$\begin{aligned} v(x,t)=&\frac{B}{\rho A}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{EI}{\rho A}\right)^r f^{(4r)}t^{2(r+1)}\\ &\sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j\frac{(j+r)!\,t^{(2-\alpha)j}}{j!\,\Gamma((2-\alpha)j+2r+3)}.\end{aligned} \tag{22}$$

Equation (21) can now be rewritten as follows

$$\begin{aligned} v(x,t)=&\frac{B}{\rho A}\\ &\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\left(\frac{EI}{\rho A}\right)^r f^{(4r)}t^{2(r+1)}E^r_{2-\alpha,\alpha r+2}\left(\frac{-c}{\rho A}t^{2-\alpha}\right).\end{aligned} \tag{23}$$

In Eq. (22), $E^r_{\lambda,\mu}(y)$ is called the Mittage-Leffler function of two parameters λ and μ. Here

$$E^r_{\lambda,\mu}(y)=\sum_{j=0}^{\infty}\frac{(j+r)!\,y^j}{j!\,\Gamma(\lambda j+\lambda r+\mu)},\quad r=0,1,2,\ldots,$$

$\lambda=2-\alpha$ and $\mu=\alpha r+2$.

It is worth mentioning that the normal mode and the Laplace transform techniques have been applied to the system (1) with $\alpha=1/2$ by Agrawal [23] to find an analytical solution. Also, the Adomain decomposition method [19] has been used to find an analytical solution. Zu-feng and Xiao-yan [19] reported in their remarks that the results obtained are identical with [23] for the special values of α and $F(x,t)$. One may find the solution of Equation (1) under homogeneous initial conditions as mentioned in [19, 23] as

$$v(x,t)=\sum_{j=1}^{\infty}\frac{1}{m_j}\phi_j\int_0^t G_j(t-\xi)f_j(\xi)d\xi, \tag{24}$$

where G_j is the fractional Green's function associated with the operator $P_j\left(\frac{d^\alpha}{dt^\alpha}\right)$, $f_j\equiv f_j(t)$ is defined by

$$f_j=\int_0^L F(x,t)\phi_j(x)dx$$

and $\phi_j\equiv\phi_j(x)$ satisfies $EI\frac{d^4\phi_j}{dx^4}=\rho A\omega_j^2\phi_j$ with the orthogonality condition

$$\int_0^L \rho A\phi_i\phi_j dx=\begin{cases} m_j, i=j\\ 0, i\neq j\end{cases}.$$

Here L is the length of the beam, m_j is the generalized mass in the j-th mode and ω_j is the natural frequency of the j-th mode. We have investigated the problem with some values of α and $F(x,t)$ below to compare the solution of [19, 23] and these agree well.

5. Numerical results and discussions

As discussed above, a unit step function response has been considered for analysis. The calculated results are depicted in plots are discussed below.

Equation (21) or (22) provides the desired expressions for the considered loading condition. In order to show the response in a precise way, some numerical results are presented in this section. We have considered a simply supported beam, hence one may have the expression for the force distribution for single degree freedom idealization as

$$f(x) = \sin(\tfrac{\pi x}{L}).$$

Here the numerical computation has been done by truncating the infinite series (21) or (22) to a finite number of terms. For numerical simulations, let us denote c/m and $EI/\rho A$ respectively as $2\eta\omega^{3/2}$ and ω^2 where, ω is the natural frequency and η is the damping ratio. The values of the parameters are taken as $B = 1$, $\rho A = 1$, $L = \pi$ and $m = 1$.

Figure (1) gives the effect of displacement against time for various values of $\alpha(= 0.2, 0.5, 0.8)$ In this computation x and η are taken as 1/2. Figures 1(a) and 11(b) present the plot for $\omega = 5rad/s$ and $\omega = 10rad/s$ respectively. A similar simulation has been done with damping ratio $\eta = 0.05$ and the obtained results are depicted in Figure 2. The dynamic responses versus time for different values of $\eta(= 0.05, 0.5, 1)$ are given in Figure 3. In this computation $\alpha = 0.2$ and $x = 1/2$ are considered. Again Figures 2(a) and 2(b) depict the plot for $\omega = 5rad/s$ and $\omega = 10rad/s$ respectively. Finally Figure 4 cites the results as above with $\alpha = 0.5$.

It is interesting to note from Figures (1) and (2) that if we increase the order of the fractional derivative α, the beam suffers more oscillations for smaller value of α. Similar observations may be made by keeping the order of the fractional derivative constant and varying the damping ratios as shown in Figures (3) and (4). It can clearly be seen that increasing the value of the damping ratios decreases the oscillations.

6. Conclusions

The Homotopy perturbation method has successfully been applied to the solution of a fractionally damped viscoelastic beam, where the fractional derivative is considered as of arbitrary order. Unit step response functions with homogeneous initial conditions are chosen to illustrate the proposed method. Performance of this method is shown and its results are compared with analytical solution obtained by Zu-feng and Xiao-yan [19]. It is interesting to note that the results obtained by the presented method exactly matches the analytical solution obtained in [19]. Though the solution by HPM is of the form of an infinite series, it can be written in a closed form in some cases. The main advantage of HPM is the capability to achieve exact solution and rapid convergences with few terms.

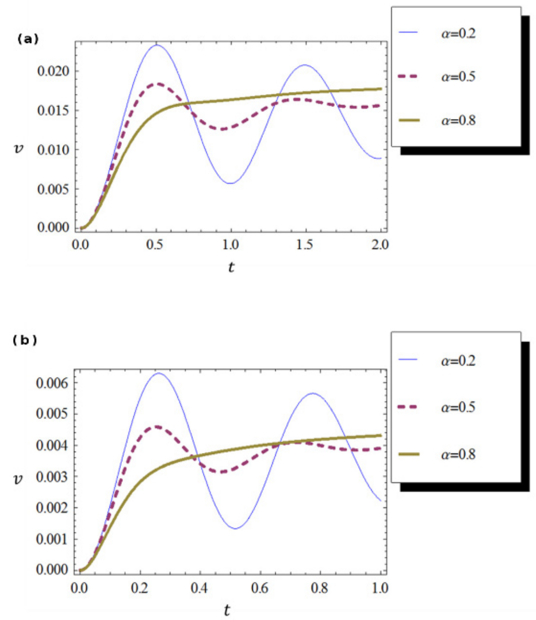

Figure 1. Unit step responses along $x = 1/2$ with natural frequency (a) $\omega = 5rad/s$, (b) $\omega = 10rad/s$ and damping ratio $\eta = 0.5$

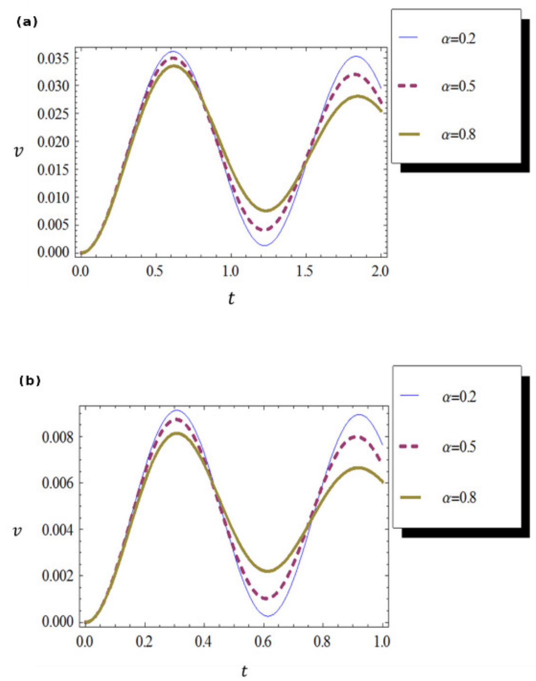

Figure 2. Unit step responses along $x = 1/2$ with natural frequency (a) $\omega = 5rad/s$, (b) $\omega = 10rad/s$ and damping ratio $\eta = 0.05$

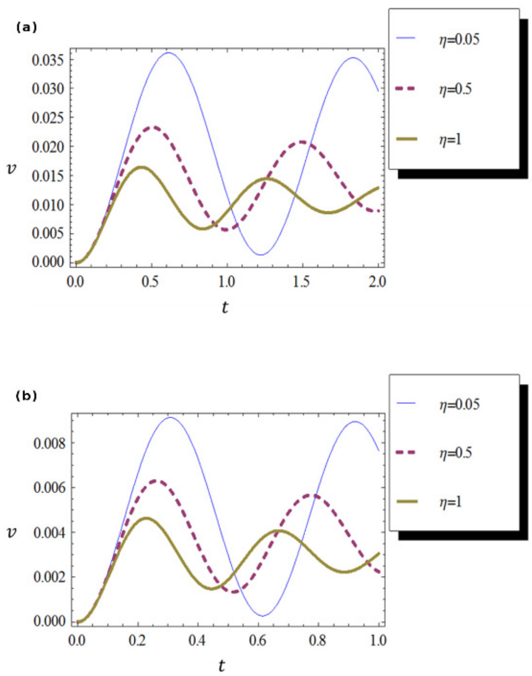

Figure 3. Unit step responses along $x = 1/2$ with natural frequency (a) $\omega = 5rad/s$, (b) $\omega = 10rad/s$ and damping ratios $\eta = 0.05,\ 0.5$ and 1 for $\alpha = 0.2$

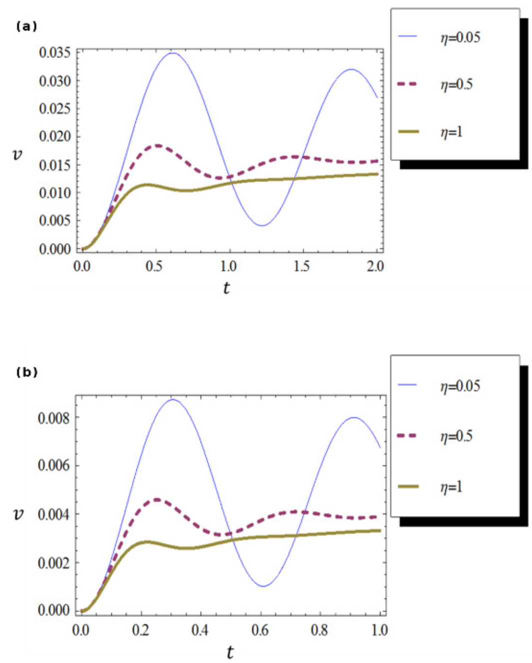

Figure 4. Unit step responses along $x = 1/2$ with natural frequency (a)$\omega = 5rad/s$, (b)$\omega = 10rad/s$ and damping ratios $\eta = 0.05,\ 0.5$ and 1 for $\alpha = 0.5$

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