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Application of Homotopy Perturbation Method and Variational Iteration Method to Nonlinear Oscillator Differential Equations

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Abstract In this paper, homotopy perturbation method (HPM) and variational iteration method (VIM) are applied to solve nonlinear oscillator differential equations. Illustrative examples reveal that these methods are very effective and convenient for solving nonlinear differential equations. Moreover, the methods do not require linearization or small perturbation. Comparisons are also made between the exact solutions and the results of the homotopy perturbation method and variational iteration method in order to prove the precision of the results obtained from both methods mentioned.

Keywords Homotopy perturbation method (HPM) · Variational iteration method (VIM) · Nonlinear oscillators · Exact solution · Van Der Pol oscillator problem

1 Introduction

This paper considers two examples of a general oscillator differential equation of the form:

$$u'' + f(t, u, u') = 0. \tag{1}$$

Subject to $u(0) = a$ and $u'(0) = b$, where t is time, u is the displacement, and the prime denotes differentiation with respect to t.

The first example involves an oscillator in which the voltage goes through a cycle and repeats its value after some time called the period of oscillation. One may think of these states as stable in the sense that slight changes in the voltage will still lead us back to the

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same place in the circuit operation. A prototype of this circuit is the van der pol circuit [1]. Another example is a nonlinear oscillator differential equation.

Balthazar Van Der Pol was a Dutch electrical engineer who initiated modern experimental dynamics in the laboratory during the 1920s and 1930s. Van Der Pol investigated electrical circuits employing vacuum tubes and found that they have stable oscillations, now known as limit cycles.

The Van Der Pol equation is:

$$\frac{d^2u}{dt^2} - k(1-u^2)\frac{du}{dt} + u = 0, \tag{2}$$

where k controls the way in which voltage flows through the system.

The Van Der Pol oscillator is one of the systems whose damping forces are nonlinear. These nonlinear damping forces have a very important property: the damping force will tend to increase the amplitude for small velocities but to decease it for large velocities.

It follows that the state of rest is not stable and that an oscillation will be built up from rest even in the absence of external forces, this accounts for the description of these oscillations as self-excited of self-sustained oscillations.

In general, the limit cycle system appears in various problems in nonlinear dynamics, particularly in the analysis of relaxation oscillation, as the Van Der Pol equation (2), limit cycle oscillators are also useful as phenomenological models for studies of the low-dimensional dynamics of the heart, In chemical kinetics and in electronics, there are certain negative-resistance oscillators [2].

A question that has been addressed in many of these investigations refers to the behavior of limit cycle system under an external, periodic force. If the phase space of the unforced system is two- dimensional, a time dependent external force can lead to different responses, like periodic, quasi-periodic and chaotic motion.

The Van Der Pol oscillator under an external periodic force is:

$$\frac{d^2u}{dt^2} - k(1-u^2)\frac{du}{dt} + u = k\beta\lambda\cos\lambda t, \tag{3}$$

where λ and $k\beta\lambda$ are the frequency and the amplitude of the external force, respectively [2].

Over the last decades several analytical/approximate methods have been developed to solve nonlinear ordinary and partial differential equations. Some of these techniques include variational iteration method (VIM) [3–8], decomposition method [9–11], homotopy perturbation method (HPM) [12–17], etc.

Linear and Nonlinear phenomena are of fundamental importance in various fields of science and engineering. Most models of real—life problems, however, are still very difficult to solve. Therefore, approximate analytical solutions such as homotopy-perturbation method [12–17] were introduced.

This method is most effective and convenient for both linear and nonlinear equations. A shortcoming of the perturbation method is that the method is based on assuming a small parameter while the majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all. Therefore the approximate solutions obtained by the perturbation methods, in most cases, are valid only for small values of the small parameter.

Generally, the perturbation solutions are uniformly valid as long as a scientific system parameter is small. However, we cannot rely fully on the approximations, because there is no criterion on which the small parameter should exists. Thus, it is essential to check the validity of the approximations numerically and/or experimentally. To overcome these difficulties, HPM have been proposed recently.

Another approximate analytical method was introduced by He [3–5] who proposed a variational iteration method based on the use of restricted variations and correction functionals which has found a wide application for the solution of nonlinear ordinary and partial differential equations. This method does not require the presence of small parameters in the differential equation, and provides the solution (or an approximation to it) as a sequence of iterates. The method does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives.

Extensive studies have been done on investigation of oscillator problems by approximate analytical methods. For example, Ganji et al. [18] applied VIM to nonlinear oscillators with discontinuities. They showed that the obtained solutions are valid for the whole domain. Furthermore, they concluded that discontinuous function had no tangible effect on the efficiency of the method. He and Wu [19] surveyed Major applications to nonlinear wave equation, nonlinear fractional differential equations, nonlinear oscillations and nonlinear problems arising in various engineering applications.

In this paper we will apply the homotopy perturbation method and variational iteration method to nonlinear oscillator differential equations.

2 Basic Idea of Homotopy-Perturbation Method

To explain this method, let us consider the following function:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{4}$$

with the boundary conditions of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma, \tag{5}$$

where $A, B, f(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively.

Generally speaking the operator A can be divided in to a linear part L and a nonlinear part $N(u)$. Equation (4) can therefore, be written as:

$$L(u) + N(u) - f(r) = 0. \tag{6}$$

By the homotopy technique, we construct a homotopy $v(r, p) : \Omega \times [0, 1] \rightarrow R$. Which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0, 1], r \in \Omega, \tag{7}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{8}$$

where $p \in [0, 1]$ is an embedding parameter, while u_0 is an initial approximation of (2), which satisfies the boundary conditions. Obviously, from (7) and (8) we will have:

$$H(v, 0) = L(v) - L(u_0) = 0, \tag{9}$$

$$H(v, 1) = A(v) - f(r) = 0. \tag{10}$$

The changing process of p from zero to unity is just that of $v(r, p)$ from u_0 to $u(r)$. In topology, this is called deformation, while $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy. According to the HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solutions of (7) and (8) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots. \tag{11}$$

Setting $p = 1$ yields in the approximate solution of (4) to:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{12}$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantage.

The series (12) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(v)$. Moreover, He made the following suggestions [14]:

- The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large, i.e. $p \to 1$.
- The norm of $L^{-1} \frac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

3 Basic Idea of Variational Iteration Method

To clarify the basic ideas of VIM [3–8], we consider the following differential equation:

$$Lu + Nu = g(t), \tag{13}$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a homogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau))d\tau \tag{14}$$

where λ is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript n indicates the nth approximation and u_n is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

4 Example 1

Consider the Van Der Pol Oscillator problem [20]:

$$\frac{d^2u}{dt^2} + \frac{du}{dt} + u + u^2\frac{du}{dt} = 2\cos t - \cos^3 t \tag{15}$$

with the initial conditions:

$$\begin{aligned} u(0) &= 0, \\ u'(0) &= 1. \end{aligned} \tag{16}$$

The exact solution of the above differential system is:

$$u(t) = \sin(t). \tag{17}$$

4.1 Application of Homotopy-Perturbation Method

To solve (15) by means of HPM, we consider the following process after separating the linear and nonlinear parts of the equation.

A homotopy can be constructed as follows:

$$H(v,p)=(1-p)\left(\left(\frac{d^2v(t)}{dt^2}+\frac{dv(t)}{dt}+v(t)\right)-1\right)$$
$$+p\left(\frac{d^2v(t)}{dt^2}+\left(\frac{d}{dt}v(t)\right)+v(t)+v(t)^2\left(\frac{d}{dt}v(t)\right)-2\cos(t)+\cos(t)^3\right) \quad (18)$$

Substituting $v=v_0+pv_1+\cdots$ in to (18) and rearranging the resultant equation based on powers of p-terms, one has:

$$p^0:-1+\left(\frac{d^2}{dt^2}v_0(t)\right)+v_0(t)+\left(\frac{d}{dt}v_0(t)\right)=0, \quad (19)$$

$$p^1:\left(\frac{dv_1(t)}{dt}\right)+1+\cos^3(t)+v_1(t)+\left(\frac{d^2v_1(t)}{dt^2}\right)-2\cos(t)+v_0^2(t)\left(\frac{dv_0(t)}{dt}\right)=0, \quad (20)$$

with the following conditions:

$$\begin{aligned}&v_0(0)=0, \qquad \frac{d}{dt}v_0(0)=1,\\ &v_i(0)=0, \qquad \frac{d}{dt}v_i(0)=0, \quad i=1,2,\ldots.\end{aligned} \quad (21)$$

With the effective initial approximation for v_0 from the conditions (21) and solutions of (19), (20) may be written as follows:

$$v_0(t)=\frac{1}{3}e^{(-0.5t)}\sin(0.5\sqrt{3}t)\sqrt{3}-e^{(-0.5t)}\cos(0.5\sqrt{3}t)+1, \quad (22)$$

$$\begin{aligned}v_1(t)=&-\frac{46961}{119574}e^{(-0.5t)}\sin(0.5\sqrt{3}t)\sqrt{3}+\frac{45263}{39858}e^{(-0.5t)}\cos(0.5\sqrt{3}t)\\&-\frac{4}{21}e^{(-2.5t)}\left(\left(\frac{7}{4}-\frac{7}{4}t\right)(e^t)^2+\frac{7}{8}e^t\cos(0.5\sqrt{3}t)+\frac{7}{4}\sqrt{3}\left(-0.5+\left(\frac{1}{3}+x\right)e^t\right)\right.\\&\times e^t\sin(0.5\sqrt{3}t)-\frac{7}{156}\sqrt{3}\sin(1.5\sqrt{3}t)e^t-\frac{7}{26}\cos(1.5\sqrt{3}t)e^t\\&+\left(-\frac{105}{16}\sin t+\frac{21}{4}-\frac{21}{146}\cos(3t)+\frac{63}{1168}\sin(3t)\right)e^{(2.5t)}+(\sqrt{3}\sin(\sqrt{3}t)\\&\left.-\frac{7}{2}+2\cos(\sqrt{3}t)e^{(1.5t)})\right).\end{aligned} \quad (23)$$

In the same manner, the rest of components were obtained using the Maple package. According to the HPM, we can conclude that:

$$u(t)=\lim_{p\to 1}v(t)=v_0(t)+v_1(t)+\cdots. \quad (24)$$

Table 1 Comparison of the approximate solutions with exact solution

t	Exact solution	HPM	VIM	Error of HPM	Error of VIM
0.0	0.0000000000	0.0000000000	0.0000000000	0.00000000E+0	0.00000000E+0
0.1	0.0998334166	0.0998334161	0.0998333893	5.5000000E−10	2.73500000E−8
0.2	0.1986693308	0.1986692604	0.1986676148	7.04000000E−8	1.71600000E−6
0.3	0.2955202067	0.2955190492	0.2955011070	1.15750000E−6	1.90997000E−5
0.4	0.3894183423	0.3894101010	0.3893138342	8.24130000E−6	1.04508100E−4
0.5	0.4794255386	0.4793884172	0.4790386507	3.71214000E−5	3.86887900E−4
0.6	0.5646424734	0.5645175388	0.563525292	1.24934600E−4	1.11718070E−3
0.7	0.6442176872	0.6438744431	0.6415028932	3.43244100E−4	2.71479400E−3
0.8	0.7173560909	0.7165444751	0.7115469778	8.11615800E−4	5.80911310E−3
0.9	0.7833269096	0.7816179472	0.7720563532	1.70896240E−3	1.12705564E−2
1.0	0.8414709848	0.8381911175	0.8212443348	3.27986730E−3	2.02266500E−2

Therefore, substituting the values of $v_0(t)$, $v_1(t)$ from (22), (23) in to (24) yields:

$$\begin{aligned}
u(t) = {} & \frac{1}{3}e^{(-0.5t)}\sin(0.5\sqrt{3}t)\sqrt{3} - e^{(-0.5t)}\cos(0.5\sqrt{3}t) + 1 \\
& - \frac{46961}{119574}e^{(-0.5t)}\sin(0.5\sqrt{3}t)\sqrt{3} + \frac{45263}{39858}e^{(-0.5t)}\cos(0.5\sqrt{3}t) \\
& - \frac{4}{21}e^{(-2.5t)}\left(\left(\frac{7}{4} - \frac{7}{4}t\right)(e^t)^2 + \frac{7}{8}e^t\cos(0.5\sqrt{3}t) + \frac{7}{4}\sqrt{3}\left(-0.5 + \left(\frac{1}{3} + x\right)e^t\right)\right. \\
& \times e^t\sin(0.5\sqrt{3}t) - \frac{7}{156}\sqrt{3}\sin(1.5\sqrt{3}t)e^t - \frac{7}{26}\cos(1.5\sqrt{3}t)e^t \\
& + \left(-\frac{105}{16}\sin t + \frac{21}{4} - \frac{21}{146}\cos(3t) + \frac{63}{1168}\sin(3t)\right)e^{(2.5t)} + (\sqrt{3}\sin(\sqrt{3}t) \\
& \left. - \frac{7}{2} + 2\cos(\sqrt{3}t)e^{(1.5t)})\right).
\end{aligned} \tag{25}$$

The solutions obtained following the above calculations using HPM are tabulated in Table 1. Comparison of the results obtained from HPM and those of the exact solution clearly reveal the high accuracy of the calculations of HPM.

4.2 Application of Variational Iteration Method

To solve (15) via VIM, one has to find the Lagrangian multiplier, which can be identified by substituting (15) into (14), upon making it stationary leads to the following:

$$\lambda''(\tau) - \lambda'(\tau) + \lambda(\tau) = 0, \tag{26a}$$

$$1 - \lambda'(\tau) + \lambda(\tau)|_{\tau=t} = 0, \tag{26b}$$

$$\lambda(\tau)|_{\tau=t} = 0. \tag{26c}$$

Solving the system of (26), yields:

$$\lambda(\tau)=\frac{2\sqrt{3}}{3}e^{(\frac{1}{2}\tau-\frac{1}{2}t)}\sin\left(\frac{\sqrt{3}}{2}(\tau-t)\right), \tag{27}$$

and the variational iteration formula is obtained in the form:

$$\begin{aligned}u_{n+1}(t)=u_n(t)+\int_0^t\Bigg(&\frac{2\sqrt{3}}{3}e^{(\frac{1}{2}\tau-\frac{1}{2}t)}\sin\left(\frac{\sqrt{3}}{2}(\tau-t)\right)\\ &\times(u''(\tau)+u'(\tau)+u(\tau)+u^2(\tau)u'(\tau)-2\cos\tau+\cos^3\tau)\Bigg)d\tau.\end{aligned} \tag{28}$$

Now, we assume that the initial approximation has the form:

$$u_0=at+b, \tag{29}$$

where a and b are unknown constants to be further determined.

By the iteration formula (28), we can directly obtain other components as:

$$\begin{aligned}u_1(t)=&-a^3t^2+(-2a^2b+2a^3)t-\frac{2}{73}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)+\frac{2\sqrt{3}}{3}a^2be^{(-\frac{1}{2}t)}\sin\left(\frac{\sqrt{3}}{2}t\right)\\ &-2a^2be^{(-\frac{1}{2}t)}\cos\left(\frac{\sqrt{3}}{2}t\right)+\frac{\sqrt{3}}{3}ab^2e^{(-\frac{1}{2}t)}\sin\left(\frac{\sqrt{3}}{2}t\right)+ab^2e^{(-\frac{1}{2}t)}\cos\left(\frac{\sqrt{3}}{2}t\right)\\ &-\frac{60\sqrt{3}}{73}e^{(-\frac{1}{2}t)}\sin\left(\frac{\sqrt{3}}{2}t\right)+\frac{\sqrt{3}}{3}be^{(-\frac{1}{2}t)}\sin\left(\frac{\sqrt{3}}{2}t\right)+\frac{2\sqrt{3}}{3}ae^{(-\frac{1}{2}t)}\sin\left(\frac{\sqrt{3}}{2}t\right)\\ &-\frac{4\sqrt{3}}{3}a^3e^{(-\frac{1}{2}t)}\sin\left(\frac{\sqrt{3}}{2}t\right)+be^{(-\frac{1}{2}t)}\cos\left(\frac{\sqrt{3}}{2}t\right)+\frac{92}{73}\sin(t)+\frac{8}{73}\cos^3(t)\\ &-\frac{6}{73}\cos(t)-\frac{3}{73}\sin(t)\cos^2(t)+2a^2b-ab^2.\end{aligned} \tag{30}$$

Incorporating the initial conditions, (16), into $u_1(t)$, we obtain:

$$a=1,\qquad b=0. \tag{31}$$

Therefore, we obtain the following first-order approximate solution:

$$\begin{aligned}u_1(t)=&-t^2+2t-\frac{2}{73}e^{(-\frac{1}{2}t)}\cos\left(\frac{\sqrt{3}}{2}t\right)-\frac{326\sqrt{3}}{219}e^{(-\frac{1}{2}t)}\sin\left(\frac{\sqrt{3}}{2}t\right)\\ &+\frac{92}{73}\sin(t)+\frac{8}{73}\cos^3(t)-\frac{6}{73}\cos(t)-\frac{3}{73}\sin(t)\cos^2(t).\end{aligned} \tag{32}$$

The obtained solution is of remarkable accuracy, as shown in Table 1.

5 Example 2

Consider the nonlinear oscillator differential equation [21]:

$$\frac{d^2u}{dt^2}-u+u^2+\left(\frac{du}{dt}\right)^2-1=0, \tag{33}$$

with the initial conditions:

$$u(0) = 2,$$
$$u'(0) = 0. \tag{34}$$

The exact solution of the above differential system is:

$$u(t) = 1 + \cos(t). \tag{35}$$

5.1 Application of Homotopy Perturbation Method

To solve (33) by means of HPM, we consider the following process after separating the linear and nonlinear parts of the equation.

A homotopy can be constructed as follows:

$$H(v, p) = (1-p)\left(\frac{d^2 v(t)}{dt^2} - \frac{d^2 v_0(t)}{dt^2}\right) + p\left(\frac{d^2 v(t)}{dt^2} + \left(\frac{d}{dt}v(t)\right)^2 + v(t)^2 - v(t) - 1\right) = 0. \tag{36}$$

Substituting $v = v_0 + pv_1 + \cdots$ in to (36) and rearranging the resultant equation based on powers of p-terms, one has:

$$p^0 : \frac{d^2}{dt^2}v_0 = 0, \tag{37}$$

$$p^1 : -1 + \left(\frac{d^2}{dt^2}v_1(t)\right) - v_0(t) + \left(\frac{d}{dt}v_0(t)\right)^2 + v_0(t)^2 = 0, \tag{38}$$

$$p^2 : 2\left(\frac{d}{dt}v_0(t)\right)\left(\frac{d}{dt}v_1(t)\right) + \left(\frac{d^2}{dt^2}v_2(t)\right) - v_1(t) + 2v_0(t)v_1(t) = 0, \tag{39}$$

with the following conditions:

$$v_0(0) = 2, \qquad \frac{d}{dt}v_0(0) = 0,$$
$$v_i(0) = 0, \qquad \frac{d}{dt}v_i(0) = 0, \quad i = 1, 2, \ldots. \tag{40}$$

With the effective initial approximation for v_0 from the conditions (40) and solutions of (37), (38) and (39) may be written as follows:

$$v_0(t) = 2, \tag{41}$$

$$v_1(t) = -0.5t^2, \tag{42}$$

$$v_2(t) = \frac{1}{8}t^4. \tag{43}$$

In the same manner, the rest of components were obtained using the Maple package. According to the HPM, we can conclude that:

$$u(t) = \lim_{p\to 1} v(t) = v_0(t) + v_1(t) + \cdots. \tag{44}$$

Therefore, substituting the values of $v_0(t)$, $v_1(t)$ and $v_2(t)$ from (41), (42) and (43) in to (44) yields:

$$u(t) = 2 - 0.5t^2 + \frac{1}{8}t^4. \tag{45}$$

As it can be seen, using HPM in solving this equation leads to the exact solution (Table 2).

5.2 Application of Variational Iteration Method

To solve (33) via VIM, one has to find the Lagrangian multiplier, which can be identified by substituting (33) into (14), upon making it stationary leads to the following:

$$\lambda''(\tau) - \lambda(\tau) = 0, \tag{46a}$$

$$1 - \lambda'(\tau)|_{\tau=t} = 0, \tag{46b}$$

$$\lambda(\tau)|_{\tau=t} = 0, \tag{46c}$$

Solving the system of equations (46), yields:

$$\lambda(\tau) = \frac{1}{2}e^{(\tau-t)} - \frac{1}{2}e^{(t-\tau)}, \tag{47}$$

and the variational iteration formula is obtained in the form:

$$u_{n+1}(t) = u_n(t) + \int_0^t \left\{ \left(\frac{1}{2}e^{(\tau-t)} - \frac{1}{2}e^{(t-\tau)} \right) (u''(\tau) - u(\tau) + u^2(\tau) + u'^2(\tau) - 1) \right\} d\tau. \tag{48}$$

Now, we assume that the initial approximation has the form:

$$u_0 = at + b, \tag{49}$$

where a and b are unknown constants to be further determined.

By the iteration formula (48), we can directly obtain other components as:

$$\begin{aligned} u_1(t) = {} & \frac{1}{2}e^t + \frac{1}{2}e^{-t} + \frac{a}{2}e^t - \frac{a}{2}e^{-t} + \frac{b}{2}e^t + \frac{b}{2}e^{-t} - \frac{3}{2}a^2e^t - \frac{3}{2}a^2e^{-t} \\ & - abe^t + abe^{-t} - \frac{b^2}{2}e^t - \frac{b^2}{2}e^{-t} + a^2t^2 + 3a^2 + 2abt + b^2 - 1. \end{aligned} \tag{50}$$

Incorporating the initial conditions, (34), into $u_1(t)$, we obtain:

$$a = 0, \qquad b = 2. \tag{51}$$

Therefore, we obtain the following first-order approximate solution:

$$u_1(t) = -\frac{1}{2}e^t - \frac{1}{2}e^{-t} + 3. \tag{52}$$

The obtained solution is of remarkable accuracy, as shown in Table 2.

Table 2 Comparison of the approximate solutions with exact solution

t	Exact solution	HPM	VIM	Error of HPM	Error of VIM
0.0	2.000000000	2.000000000	2.000000000	0.0000000E+0	0.0000000E+0
0.1	1.995004165	1.995012500	1.994995832	8.3350000E−6	8.3330000E−6
0.2	1.980066578	1.980200000	1.979933244	1.3342200E−4	1.3333400E−4
0.3	1.955336489	1.956012500	1.954661486	6.7601100E−4	6.7500300E−4
0.4	1.921060994	1.923200000	1.918927628	2.1390060E−3	2.1333660E−3
0.5	1.877582562	1.882812500	1.872374035	5.2299380E−3	5.2085270E−3
0.6	1.825335615	1.836200000	1.814534782	1.0864385E−2	1.0800833E−2
0.7	1.764842187	1.785012500	1.744830994	2.0170313E−2	2.0011193E−2
0.8	1.696706709	1.731200000	1.662565054	3.4493291E−2	3.4141655E−2
0.9	1.621609968	1.677012500	1.566913614	5.5402532E−2	5.4696354E−2
1.0	1.540302306	1.625000000	1.456919365	8.4697694E−2	8.3382941E−2

6 Conclusions

Homotopy perturbation method and variational iteration method are employed successfully to study nonlinear oscillator differential equations. In conclusion, HPM and VIM provide highly accurate numerical solutions for nonlinear problems in comparison with other methods. As it is mentioned, these methods avoid linearization and physically unrealistic assumptions.

Finally, Comparison with exact solution reveals that homotopy perturbation method and variational iteration method are remarkably effective for solving boundary value problems. For Computations and plots, Maple Package has been used.

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