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AN ANALYTICAL APPROACH TO LARGE AMPLITUDE VIBRATION AND POST-BUCKLING OF FUNCTIONALLY GRADED BEAMS REST ON NON-LINEAR ELASTIC FOUNDATION

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In this paper, non-linear vibration and post-buckling analysis of beams made of functionally graded materials (FGMs) rest on a non-linear elastic foundation subjected to an axial force are studied. Based on Euler-Bernoulli beam theory and von-Karman geometric non-linearity, the partial differential equation (PDE) of motion is derived.Then, this PDE problem is simplified into an ordinary differential equation problem by using the Galerkin method. Finally, the governing equation is analytically solved using the variational iteration method (VIM). The results from the VIM solution are compared and shown to be in excellent agreement with the available solutions from the open literature. Some new results for the non-linear natural frequencies and buckling load of functionally graded (FG) beams, such as effects of vibration amplitude, elastic coefficients of foundation, axial force, end supports and material inhomogenity are presented for future references.

Key words: functionally graded beams, non-linear vibration, post-buckling, Galerkin method, variational iteration method (VIM)

1. Introduction

Recently, a new class of composite materials known as functionally graded materials (FGMs) has drawn considerable attention. Typically, FGMs are made from a mixture of metals and ceramics and are further characterized by a smooth and continuous change of mechanical properties from one surface to another. It has been reported that the weakness of the fiber reinforced laminated composite materials, such as debonding, huge residual stress, locally largely plastic deformations, etc., can be avoided or reduced in FGMs (Noda, 1991; Tanigawa, 1995). FGMs were initially designed as thermal barrier materials for aerospace structures and fusion reactors where extremely high temperature and large thermal gradient exist. With the increasing demand, FGMs have been widely used in general structures. Hence, many FGM structures have been extensively studied, such as functionally graded (FG) beams, plates, shells, etc. Furthermore, due to huge application of beams in different fields such as civil, marine and aerospace engineering, it is necessary to study their dynamic behavior at large amplitudes which is effectively non-linear and therefore, is governed by non-linear equations. Non-linear free vibrationsand buckling analysis of isotropic and composite beams have received a good amount of attention in the literature (see References).

The Galerkin finite element method has been presented for studying non-linear vibrations of beams describable in terms of the moderately large bending theory by Bhashyam and Prathap (1980). Non-linear free vibration analysis of laminated composite beams was studied by Kapania and Raciti (1989), using the perturbation method. Large amplitude free vibration of an unsymmetrically laminated beam using von-Karman large deflection theory was studied by Singh *et al.* (19991). Free non-linear and undamped vibration analysis of flexible non-prismatic Euler-Bernoulli beams was investigated by Fertis and Afonta (1992). An analytical method for determining the vibration modes of geometrically non-linear beams under various edge conditions was presented by Qaisi (1993). Nayfeh and Nayfeh (1995) obtained non-linear modes and natural frequencies of a simply supported Euler-Bernoulli beam resting on an elastic foundation with distributed quadratic and cubic non-linearities using the method of multiple scales and the invariant manifold approach. The geometrically non-linear analysis of flexible sliding beams based on the assumptions of Euler-Bernoulli beam theory was studied by Behdinan *et al.* (1997a,b). Non-linear vibrations of an Euler-Bernoulli beam with a concentrated mass attached to it were investigated by Karlik *et al.* (1998). The beam carried its own weight as well as other weights that were attached to the beam and participated in its vibrational motion. Ganapathi *et al.* (1998) studied large amplitude free vibrations of cross-ply laminated straight and curved beams using the spline element method. Patel *et al.* (1999) investigated non-linear free flexural vibrations and post-buckling of laminated orthotropic beams resting on a class of a two-parameter elastic foundation using a three-nodded shear flexible beam element. Azrar *et al.* (1999) developed a semi-analytical approach to the non-linear dynamic response problem based on Lagrange's principle and the harmonic balance method. Hatsunaga (2001) presented natural frequencies and buckling stresses of simply supported laminated composite beams taking into account the effects of transverse shear and rotary inertia. A non-linear modal analysis approach based on the invariant manifold method was utilized to obtain the non-linear normal modes of a clamped-clamped beam for large amplitude displacements by Xie *et al.* (2002). Guo and Zhong (2004) investigated non-linear vibrations of thin beams based on sextic cardinal spline functions, a spline-based differential quadrature method. Non-linear normal modes of vibration for a hinged-hinged beam with fixed ends were evaluated, considering both the continuous system and finite element models by Carlos *et al.* (2004).

In recent years, Sapountzakis and Tsiatas (2007) investigated the flexural buckling of composite Euler-Bernoulli beams of arbitrary cross sections. The resulting boundary-value problems were solved using the boundary element method. Aydogdu (2007) investigated the thermal buckling of cross-ply laminated beams with different boundary conditions. Nayfeh and Emam (2008) obtained a closed-form solution for the post-buckling configurations of beams composed of isotropic materials with various boundary conditions. Jun *et al.* (2008) investigated free vibration and buckling behavior of axially loaded laminated composite beams having arbitrary lay-up using the dynamic stiffness method taking into account the influence of axial forces, Poisson effect, axial deformation, shear deformation and rotary inertia. Pirbodaghi *et al.* (2009) used the first-order approximation of the homotopy analysis method to investigate non-linear free vibrations of Euler-Bernoulli beam. Malekzadeh *et al.* (2009) studied non-linear free vibrations of laminated composite thin beams rest on a non-linear elastic foundation (including shearing layer) with elastically restrained against rotation edges by the differential quadrature approach. Gupta *et al.* (2009) studied non-linear free vibrations of isotropic beams using a simple iterative finite element formulation. An exact solution for the post-buckling of a symmetrically laminated composite beam with fixed-fixed, fixed-hinged, and hinged-hinged boundary conditions was presented by Emam and Nayfeh (2009). Gupta *et al.* (2010a,b) recently applied the concept of coupled displacement field criteria to investigate the post-buckling behavior of isotropic (Gupta *et al.*, 2010b) and composite beams (Gupta *et al.*, 2010a). Gunda *et al.* (2010) employed the Rayleigh-Ritz method to study large amplitude vibrations of a laminated composite beam with symmetric and asymmetric lay – up orientations.

More recently, the large amplitude vibration and post-buckling analysis of FG beams have attracted increasing research efforts. Ke *et al.* (2010) used the direct numerical integration method together with Runge-Kutta technique to find the non-linear vibration response of FG beams with different end supports. Simsek (2010) studied the non-linear forced vibration of Timoshenko FG beams under action of moving harmonic load. Ma and Lee (2011) presented a further discussion of non-linear mechanical behavior for FG beams under in-plane thermal loading. Fallah and Aghdam (2011) studied large amplitude free vibrations and post-buckling of FG beams subjected to an axial force.

The pursuit of analytical solutions for the non-linear equation arising in vibration and post-buckling analysis of FG beams is of intrinsic scientific interest. The primary purpose of the present paper is to investigate an analytical solution for non-linear vibration and post-buckling of beams made of FGMs rest on a non-linear elastic foundation subjected to an axial force. Analytical expressions for non-linear natural frequencies and buckling load of FG beams are determined using the variational iteration method (VIM) given by He (1999).

2. Basic idea of variational iteration method

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x) \tag{2.1}$$

where L is a linear operator, N a non-linear operator, and $g(x)$ is the forcing term. According to the VIM, we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda[Lu_n(t) + N\widetilde{u}_n(t) - g(t)]\, dt \tag{2.2}$$

where λ is a Lagrange multiplier which can be identified optimally via the VIM. The subscripts n denote the n-th approximation, $\widetilde{u}_n$ is considered as a restricted variation, that is $\delta\widetilde{u}_n = 0$; and Eq. (2.2) is called a correction functional. The solution to linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. In this method, it is required first to optimally determine the Lagrange multiplier λ. The successive approximation u_{n+1}, $n \geqslant 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by

$$u = \lim_{n\to\infty} u_n \tag{2.3}$$

3. Problem statement

Consider a straight FG beam of length L, width b and thickness h rests on an elastic non-linear foundation and subjected to an axial force of magnitude $\overline{P}$ as shown in Fig. 1. The beam is

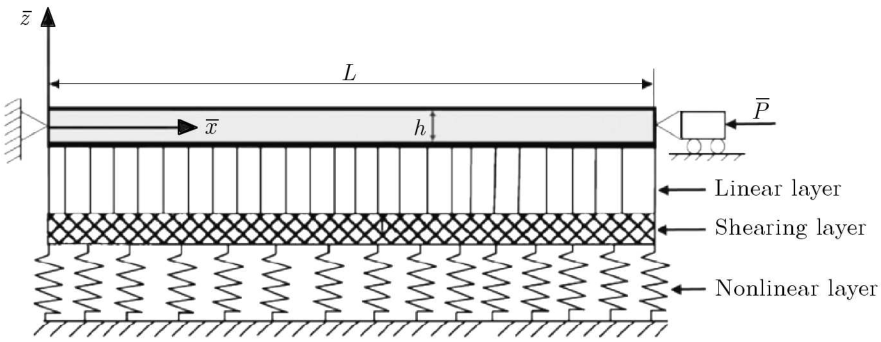

Fig. 1. Schematic of the FG beam with a non-linear foundation

supported on an elastic foundation with cubic non-linearity and shearing layer. In this study, material properties are considered to vary in accordance with the rule of mixtures as

$$P = P_M V_M + P_C V_C \tag{3.1}$$

where P and V are the material property and volume fraction, respectively, and the subscripts M and C refer to the metal and ceramic constituents, respectively. Simple power law distribution from pure metal at the bottom face $(\overline{z} = -h/2)$ to pure ceramic at the top face $(\overline{z} = +h/2)$ in terms of volume fractions of the constituents is assumed as (Ke *et al.*, 2010)

$$V_C = \Big(\frac{2\overline{z}+h}{2h}\Big)^n \qquad\qquad V_M = 1 - V_C \tag{3.2}$$

where n is the volume fraction exponent. The value of n equal to zero represents a fully ceramic beam. The mechanical and thermal properties of FGMs are determined from the volume fraction of material constituents. We assume that the non-homogeneous material properties such as the modulus of elasticity E, Poisson's ratio ν and mass density ρ can be determined by substituting Eq. (3.2) into Eq. (3.1) as

$$\begin{aligned} E(\overline{z}) &= E_M + (E_C - E_M)\Big(\frac{2\overline{z}+h}{2h}\Big)^n \qquad\qquad \nu(\overline{z}) = \nu_M + (\nu_C - \nu_M)\Big(\frac{2\overline{z}+h}{2h}\Big)^n \\ \rho(\overline{z}) &= \rho_M + (\rho_C - \rho_M)\Big(\frac{2\overline{z}+h}{2h}\Big)^n \end{aligned} \tag{3.3}$$

The force and moment resultants per unit length, based on classical theory of beams in a Cartesian coordinate system, can be written as (Emam and Nayfeh, 2009)

$$\begin{Bmatrix} N_{\overline{x}} \\ M_{\overline{x}} \end{Bmatrix} = b \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} \begin{Bmatrix} \overline{u}_{,\overline{x}} + \dfrac{1}{2}\overline{w}^2_{,\overline{x}} \\ \overline{w}_{,\overline{x}\overline{x}} \end{Bmatrix} \tag{3.4}$$

in which $\overline{w}$ and $\overline{u}$ are the transverse and axial displacements of the beam along the $\overline{z}$ and $\overline{x}$ directions, respectively. The stiffness coefficients A_{11}, B_{11} and D_{11} are given as follows

$$(A_{11}, B_{11}, D_{11}) = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(\overline{z})}{1-\nu^2(\overline{z})}(1, \overline{z}, \overline{z}^2)\, d\overline{z} \tag{3.5}$$

After some mathematical simplifications (Emam and Nayfeh, 2009), the governing equation of non-linear free vibration of an FG beam in terms of transverse displacement can be written as

$$I_1\overline{w}_{,\overline{t}\overline{t}} + b\Big(D_{11} - \frac{B_{11}^2}{A_{11}}\Big)\overline{w}_{,\overline{x}\overline{x}\overline{x}\overline{x}} + \Big(\overline{P} - \frac{bA_{11}}{2L}\int\limits_0^L \overline{w}^2_{,\overline{x}}\, d\overline{x} - \frac{bB_{11}}{L}[\overline{w}_{,\overline{x}}(L,\overline{t}) - \overline{w}_{,\overline{x}}(0,\overline{t})]\Big)\overline{w}_{,\overline{x}\overline{x}} = F_{\overline{w}} \tag{3.6}$$

in which the comma denotes the derivative with respect to $\overline{x}$ or $\overline{t}$. Furthermore, I_1 and $F_{\overline{w}}$ are the inertia term and reaction of the elastic foundation on the beam, which are defined as

$$I_1 = \int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \rho(\overline{z})\, d\overline{z} \qquad\qquad F_{\overline{w}} = -\overline{k}_L\overline{w} - \overline{k}_{NL}\overline{w}^3 + \overline{k}_S\overline{w}_{,\overline{x}\overline{x}} \tag{3.7}$$

where $\overline{k}_L$ and $\overline{k}_{NL}$ are linear and non-linear elastic foundation coefficients, respectively, and $\overline{k}_S$ is the coefficient of shear stiffness of the elastic foundation.

For convenience, in the subsequent analysis we use the following non-dimensional variables

$$x = \frac{\overline{x}}{L} \qquad w = \frac{\overline{w}}{r} \qquad t = \overline{t}\sqrt{\frac{b\Big(D_{11} - \frac{B_{11}^2}{A_{11}}\Big)}{I_1 L^4}} \tag{3.8}$$

where $r = \sqrt{I/A}$ is the radius of gyration of the cross section. Using Eqs. (3.6) and (3.7) together with the dimensionless variables defined in Eq. (3.8), the dimensionless form of the governing equation becomes

$$w_{,tt} + w_{,xxxx} + \Big(P - \frac{1}{2}\Lambda\int\limits_0^1 w_{,x}^2\;dx - B[w_{,x}(1,t) - w_{,x}(0,t)]\Big)w_{,xx} + k_L w + k_{NL}w^3 - k_S w_{,xx} = 0 \tag{3.9}$$

where

$$\begin{aligned}
&P = \frac{\overline{P}L^2}{b\Big(D_{11} - \dfrac{B_{11}^2}{A_{11}}\Big)} \qquad &&\Lambda = \frac{A_{11}r^2}{\Big(D_{11} - \dfrac{B_{11}^2}{A_{11}}\Big)} \qquad &&B = \frac{B_{11}r}{\Big(D_{11} - \dfrac{B_{11}^2}{A_{11}}\Big)}\\
&k_L = \frac{\overline{k}_L L^4}{b\Big(D_{11} - \dfrac{B_{11}^2}{A_{111}}\Big)} \qquad &&k_{NL} = \frac{\overline{k}_{NL} r^2 L^4}{b\Big(D_{11} - \dfrac{B_{11}^2}{A_{111}}\Big)} \qquad &&k_S = \frac{\overline{k}_S L^2}{b\Big(D_{11} - \dfrac{B_{11}^2}{A_{11}}\Big)}
\end{aligned} \tag{3.10}$$

Assuming $w(x,t) = V(t)\phi(x)$ where $V(t)$ is an unknown time-dependent function and $\phi(x)$ is the first eigenmode of the beam presented in Table 1, which must satisfy the kinematic boundary conditions. Applying Galerkin's method, the governing equation of motion is obtained as follows

$$\ddot{V}(t) + (\alpha_1 + P\alpha_p + \alpha_{k_L} + \alpha_{k_S})V(t) + \alpha_2 V^2(t) + (\alpha_{k_{NL}} + \alpha_3)V^3(t) = 0 \tag{3.11}$$

where

$$\begin{aligned}
&\alpha_1 = \frac{\int_0^1 \phi''''\phi\;dx}{\int_0^1 \phi^2\;dx} \qquad &&\alpha_p = \frac{\int_0^1 \phi''\phi\;dx}{\int_0^1 \phi^2\;dx} \qquad &&\alpha_2 = -B[\phi'(1) - \phi'(0)]\alpha_p\\
&\alpha_{k_L} = k_L \qquad &&\alpha_{k_{NL}} = k_{NL}\frac{\int_0^1 \phi^4\;dx}{\int_0^1 \phi^2\;dx} \qquad &&\alpha_{k_S} = -k_S\alpha_p\\
&\alpha_3 = -\Lambda\int_0^1 \phi'^2\;dx
\end{aligned} \tag{3.12}$$

The beam centroid is subjected to the following initial conditions

$$V(0) = a \qquad \frac{dV(0)}{dt} = 0 \tag{3.13}$$

where a denotes the non-dimensional maximum amplitude of oscillation. From Eq. (3.11), the post-buckling load-deflection relation of the FG beam can be obtained as

$$P_{NL} = -\frac{\alpha_1 + \alpha_{k_L} + \alpha_{k_S} + \alpha_2 V + (\alpha_{k_{NL}} + \alpha_3)V^2}{\alpha_p} \tag{3.14}$$

It should be noted that neglecting the contribution of V in Eq. (3.14), the linear buckling load can be determined as

$$P_L = -\frac{\alpha_1 + \alpha_{k_L} + \alpha_{k_S}}{\alpha_p} \tag{3.15}$$

Table 1. Trial functions for a FG beam with various boundary condition

Boundary condition	$\phi(x)$	Value of q
Simply supported	$\sin\dfrac{qx}{L}$	π
Clamped-clamped	$\Big(\cosh\dfrac{qx}{L}-\cos\dfrac{qx}{L}\Big)-\dfrac{\cosh q-\cos q}{\sinh q-\sin q}\Big(\sinh\dfrac{qx}{L}-\sin\dfrac{qx}{L}\Big)$	4.730041
Clamped-simply Supported	$\Big(\cosh\dfrac{qx}{L}-\cos\dfrac{qx}{L}\Big)-\dfrac{\cosh q-\cos q}{\sinh q-\sin q}\Big(\sinh\dfrac{qx}{L}-\sin\dfrac{qx}{L}\Big)$	3.926602

4. Implementation of VIM

Equation (3.11) can be simplified as

$$\ddot{V}+\beta_1 V+\beta_2 V^2+\beta_3 V^3=0 \tag{4.1}$$

where $\beta_1=\alpha_1+P\alpha_P+\alpha_{K_L}+\alpha_{K_S}$, $\beta_2=\alpha_2$ and $\beta_3=\alpha_{K_{NL}}+\alpha_3$. In order to solve Eq. (4.1) using VIM, we construct a correction functional, as follows

$$V_{n+1}(t)=V_n(t)+\int\limits_0^t \lambda\Big[\frac{d^2V_n(\tau)}{d\tau^2}+\omega^2V_n(\tau)+\beta_1\widetilde{V}_n(\tau)+\beta_2\widetilde{V}_n^2(\tau)+\beta_3\widetilde{V}_n^3(\tau)-\omega^2\widetilde{V}_n(\tau)\Big]\,d\tau \tag{4.2}$$

Its stationary conditions can be obtained as follows

$$\lambda''(\tau)\Big|_{\tau=t}+\omega^2\lambda(\tau)\Big|_{\tau=t}=0 \qquad 1-\lambda'(\tau)\Big|_{\tau=t}=0 \qquad \lambda(\tau)\Big|_{\tau=t}=0 \tag{4.3}$$

Thus, the Lagrangian multiplier can therefore be identified as

$$\lambda=\frac{1}{\omega}\sin[\omega(\tau-t)] \tag{4.4}$$

As a result, we obtain the following iteration formula

$$\begin{aligned}V_{n+1}(t)&=V_n(t)+\int\limits_0^t \frac{1}{\omega}\sin[\omega(\tau-t)]\\&\cdot\Big[\frac{d^2V_n(\tau)}{d\tau^2}+\omega^2V_n(\tau)+\beta_1\widetilde{V}_n(\tau)+\beta_2\widetilde{V}_n^2(\tau)+\beta_3\widetilde{V}_n^3(\tau)-\omega^2\widetilde{V}_n(\tau)\Big]\,d\tau\end{aligned} \tag{4.5}$$

From the initial conditions in Eq. (3.13), that we have it in the point $t=0$, an arbitrary initial approximation can be obtained

$$V_0(t)=a\cos(\omega t) \tag{4.6}$$

This initial approximation is a trial function, and it is used to obtain a more accurate approximate solution to Eq. (3.11). Here, ω is the non-linear frequency. Expanding the non-linear part, we have

$$N[V_0(t)]=\beta_1 a\cos(\omega t)+\beta_2[a\cos(\omega t)]^2+\beta_3[a\cos(\omega t)]^3-\omega^2 a\cos(\omega t) \tag{4.7}$$

Then

$$\begin{aligned}N[V_0(t)]&=\Big(-a\omega^2+a\beta_1+\frac{3}{4}\beta_3a^3\Big)\cos(\omega t)+\frac{1}{4}\beta_3a^3\cos(3\omega t)\\&+\frac{1}{2}\beta_2a^2\cos(2\omega t)+\frac{1}{2}\beta_2a^2\end{aligned} \tag{4.8}$$

In order to ensure that no secular terms appear in the next iteration, the coefficient of $\cos(\omega t)$ must vanish. Therefore

$$\omega = \sqrt{\beta_1 + \frac{3}{4}\beta_3 a^2} \tag{4.9}$$

Thus, the non-linear to the linear frequency ratio can be determined as

$$\frac{\omega_{NL}}{\omega_L} = \sqrt{1 + \frac{3}{4}\frac{\beta_3}{\beta_1}a^2} \tag{4.10}$$

Using the variational formula (4.5), we have

$$\begin{aligned} V_1(t) &= a\cos(\omega t) + \int\limits_0^t \frac{1}{\omega}\sin\omega(\tau - t)\Big[\frac{d^2 a\cos(\omega\tau)}{d\tau^2} + \omega^2 a\cos(\omega\tau) \\ &\quad + \frac{1}{4}\beta_3 a^3\cos(3\omega\tau) + \frac{1}{2}\beta_2 a^2\cos(2\omega\tau) + \frac{1}{2}\beta_2 a^2\Big]\, d\tau \end{aligned} \tag{4.11}$$

Then first-order approximate solution is obtained as:

$$V_1(t) = \Big(a + \frac{\beta_2 a^2}{3\omega^2} - \frac{\beta_3 a^3}{32\omega^2}\Big)\cos(\omega t) + \frac{\beta_2 a^2}{6\omega^2}\cos(2\omega t) + \frac{\beta_3 a^3}{32\omega^2}\cos(3\omega t) - \frac{\beta_2 a^2}{2\omega^2} \tag{4.12}$$

Accordingly, inserting Eq. (4.12) into Eq. (3.14), the post-buckling load-deflection can be obtained.

5. Results and discussion

In this Section, we present the results with the VIM, which was described in the previous section for solving Eq. (3.11). To test the validity and accuracy of the method used in this study, variations of the non-dimensional amplitude versus time obtained by the VIM and well-established Runge-Kutta method are displayed in Fig. 2. This figure shows a very good agreement between theVIM and numerical solution.

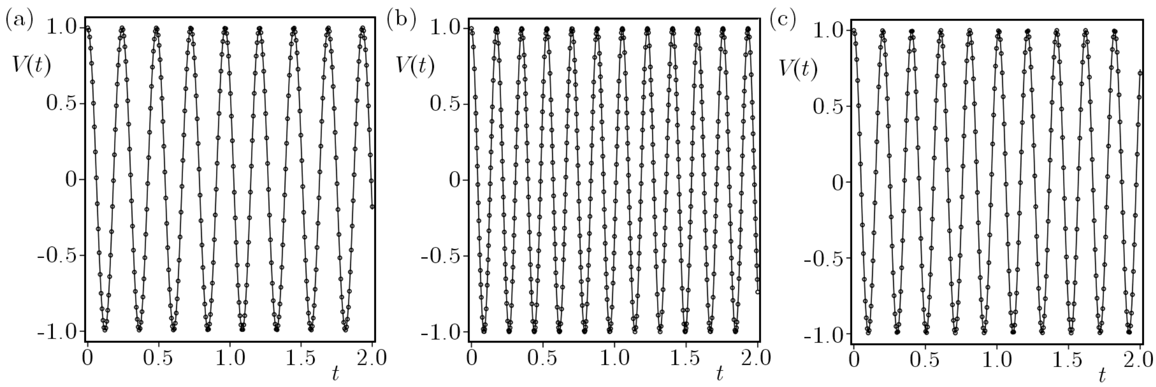

Fig. 2. Variation of the non-dimensional amplitude versus t obtained by VIM (solid line) and NS (circle): (a) simply supported, (b) clamped-clamped, (c) clamped-simply supported; ($k_S = 50$, $k_L = 50$, $k_{NL} = 50$, $P = 1$, $n = 2$, $a = 1$)

Also, to demonstrate the accuracy of the mentioned method, the non-linear to linear frequency ratio ω_{NL}/ω_L of the isotropic beams with simply supported ($\beta_1 = 97.4091$, $\beta_2 = 0$, $\beta_3 = 24.3523$) and clamped boundary conditions ($\beta_1 = 501.8177$, $\beta_2 = 0$, $\beta_3 = 76.3051$) are

Table 2. Comparison of the frequency ratio ω_{NL}/ω_L

a	SS					CC				
	Present	[29]	[2]	[28]	[7]	Present	[29]	[2]	[28]	[7]
1	1.0897	1.0897	1.0891	1.0897	1.0897	1.0554	1.0628	1.0221	1.0572	1.0552
2	1.3228	1.3229	1.3177	1.3228	1.3229	1.2067	1.2140	1.0856	1.2125	1.2056
3	1.6393	1.6394	1.6256	1.6393	1.6393	1.4235	1.3904	1.1831	1.4344	1.4214
4	2.0000	–	–	1.9999	1.9999	1.6806	1.5635	1.3064	1.6171	1.6776

compared with those reported in the previous literature in Table 2. It is observed that the present results agree very well with those given by Qaisi (1993), Azrar *et al.* (1999), Pirbodaghi *et al.* (2009), Fallah and Aghdam (2011).

The material properties presented in Table 3 are applied to next verifications and to non-linear analysis of FG beams. Table 4 shows the comparison of our solution with the published data (Gunda *et al.*, 2010; Ke *et al.*, 2010; Fallah and Aghdam, 2011) for the frequency ratio ω_{NL}/ω_L and buckling load ratio P_{NL}/P_L for both clamped-clamped (CC) and simply supported (SS) FG beams.As mentioned before, Eq. (4.1) contains a quadratic non-linearity mainly due to the bending-stretching coupling. However, for CC beams either isotropic, composite or FG, the quadratic term vanishes and, therefore, Eq. (4.1) is reduced to a Duffing equation. In this case, the exact non-linear frequency ω_{exact} can be determined as shown in Younesian *et al.* (2010). Hence, in Table 5, the frequency ratio for CC FG beam is listed. A reasonably good agreement with previous results for non-linear analysis of SS and CC FG beams can be observed in Tables 4 and 5.

Table 3. Material properties of the constituent materials of the FG beams

Properties of material	SuS304	Si_3N_4
Young's modulus E [GPa]	207.8	322.3
Poisson's ratio ν [–]	0.3178	0.24
Material density ρ [kg/m^3]	8166	2370

Table 4. Frequency ratio ω_{NL}/ω_L and buckling load ratio P_{NL}/P_L of the FG beam $(k_L = 50, k_{NL} = 10,\ k_S = 5,\ P = 2,\ n = 2)$

	a	ω_{NL}/ω_L				P_{NL}/P_L	
		Present	[10]	[20]	[7]	Present	[7]
SS	0	1.000	1.000	1.000	1.000	1.000	1.000
	0.5	1.016	1.008	1.008	1.008	1.021	1.021
	1	1.065	1.048	1.046	1.048	1.123	1.123
	1.5	1.141	1.118	1.116	1.117	1.305	1.305
	2	1.239	1.211	1.209	1.210	1.567	1.567
CC	0	1.000	1.000	1.000	1.000	1.000	1.000
	0.5	1.015	1.015	1.015	1.015	1.038	1.038
	1	1.058	1.058	1.058	1.058	1.152	1.152
	1.5	1.126	1.126	1.126	1.126	1.343	1.343
	2	1.215	1.215	1.215	1.215	1.609	1.609

Table 5. Frequency ratio ω_{NL}/ω_L of the clamped FG beam

a	$n = 0.5$		$n = 1$		$n = 2$	
	Present	Exact	Present	Exact	Present	Exact
0	1.000	1.000	1.000	1.000	1.000	1.000
1	1.059	1.059	1.059	1.059	1.058	1.058
2	1.220	1.218	1.219	1.217	1.215	1.212
3	1.449	1.441	1.447	1.439	1.439	1.431
4	1.720	1.704	1.716	1.700	1.704	1.689
5	2.015	1.990	2.010	1.986	1.994	1.971

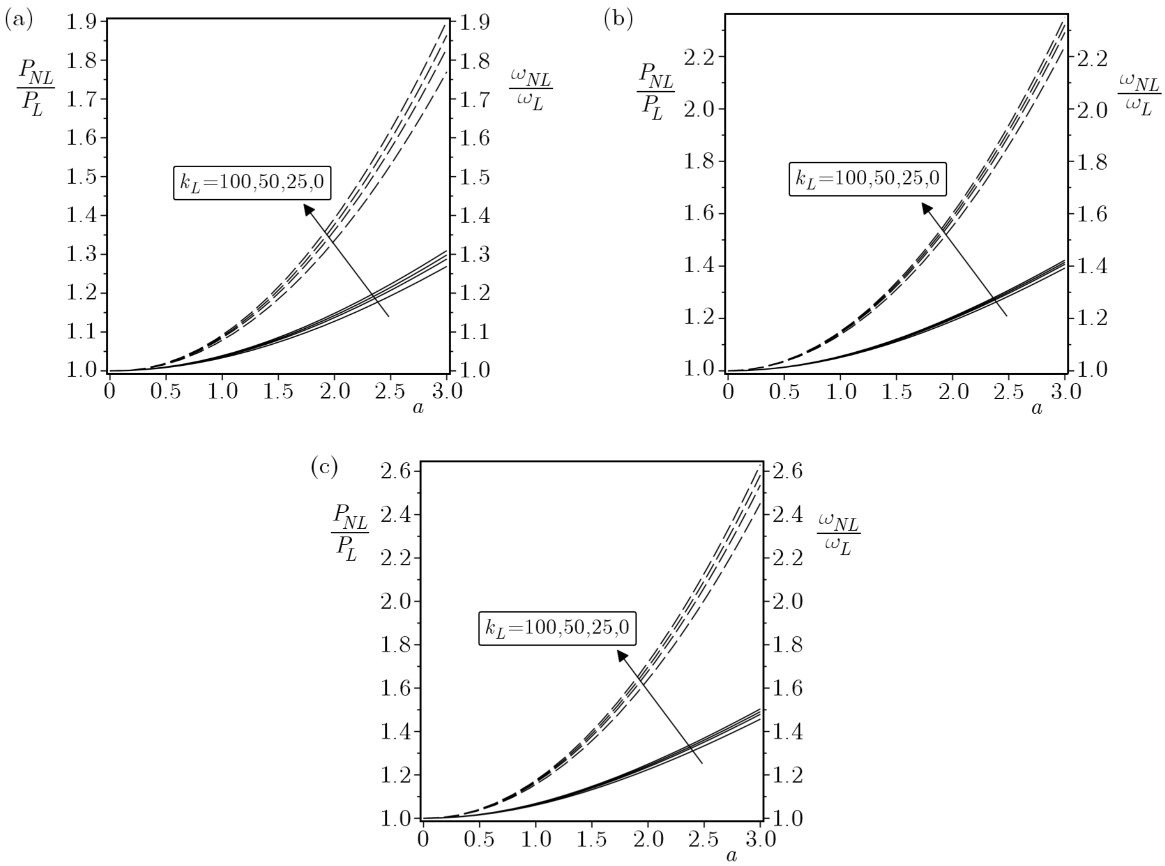

Fig. 3. Effect of the linear foundation stiffness on (a) SS FG, (b) CC FG and (c) CS FG beam for frequency (solid line) and buckling load (dash line) ratios $(k_{NL} = 50,\ k_S = 50,\ P = 1,\ n = 2)$

After these verifications, we investigate the effects of foundation parameters, axial force, vibration amplitude, end supports, such as SS, CC and clamped-simply supported (CS), and material inhomogenity on the non-linear free vibrations and post-buckling behavior of FG beams. Figures 3a,b,c demonstrate effects of the linear foundation parameter for SS, CC and CS FG beams. The curve marked $k_L = 0$ represents the case when the beam just has the shearing layer and non-linear foundation stiffness. It can be seen from these figures that all beams exhibit typical hardening behavior, i.e., the non-linear frequency ratio increases as the linear foundation parameter is decreased. Moreover, when the boundary conditions are CS, k_L induces a higher effect on the non-linear free vibrations and post-buckling behavior of FG beams.

It can be observed from Figs. 4a,b,c that an increase in the value of the shearing layer stiffness results in a decreasing hardening characteristic of the beam, i.e., a decrease in the rate of k_S causes an increase in the non-linear frequency and post-buckling strength with amplitude. The top curves represent the case when the beam just has linear and non-linear foundation stiffness. In addition, when the boundary conditions are CC, k_S induces a lower effect on the non-linear free vibrations and post-buckling behavior of FG beams. Nevertheless, an increase in the value of the non-linear foundation parameter results in increasing non-linear frequency and post-buckling strength with amplitude. This interesting behavior is shown in Figs. 5a,b,c. Also, the effects of boundary conditions together with k_{NL} on the non-linear free vibrations and post-buckling ratio may be interpreted as shown in Fig. 3.

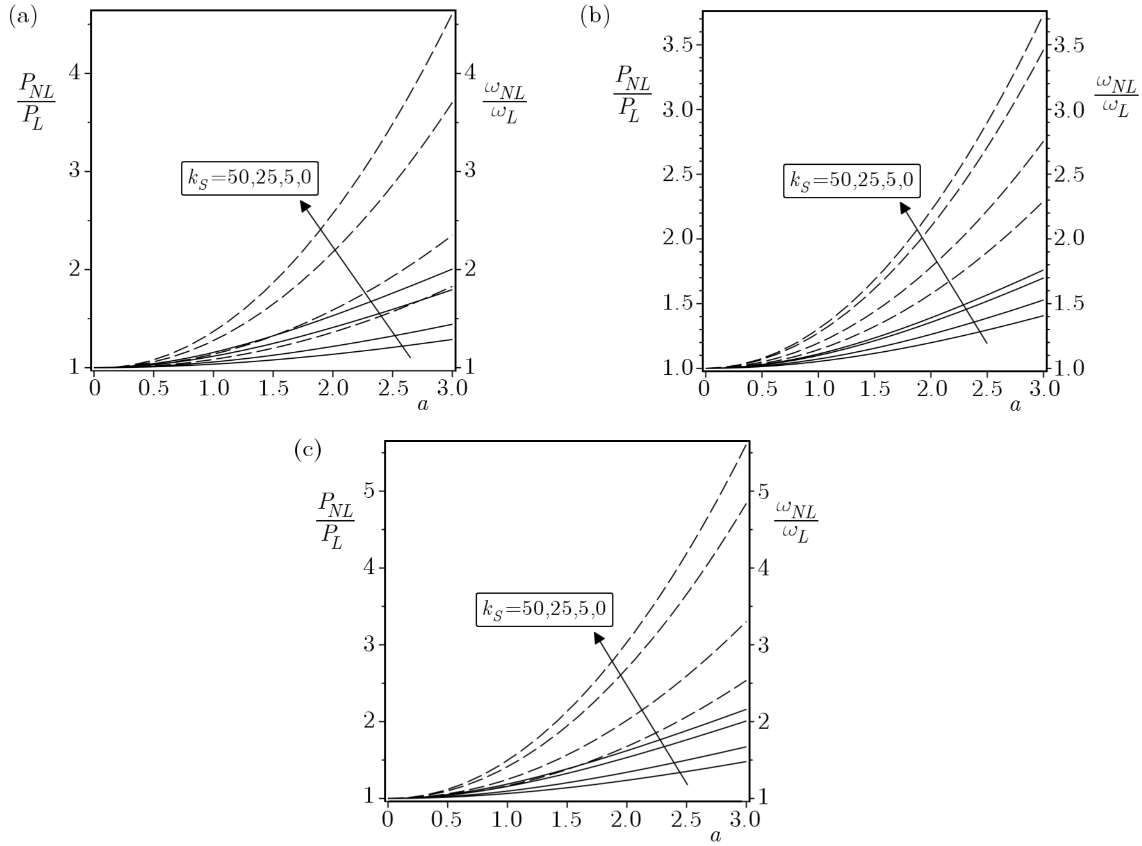

Fig. 4. Effect of the shearing layer stiffness on (a) SS FG, (b) CC FG and (c) CS FG beam for frequency solid line) and buckling load (dash line) ratios ($k_{NL} = 50$, $k_S = 50$, $P = 1$, $n = 2$)

Moreover, the effect of the axial force on the non-linear natural frequency of SS, CC and CS FG beams is presented in Figs. 6a,b,c. Results in the figures reveal that as the value of the axial load increases, the frequency ratio increases as well.

Furthermore, the influences of material inhomogenity in terms of the volume fraction exponent, axial force and dimensionless maximum amplitude on the frequency ratio are presented in Table 6. It is interesting to note from this table that the frequency ratio of the beam initially increases and then decays by increasing in the value of n.

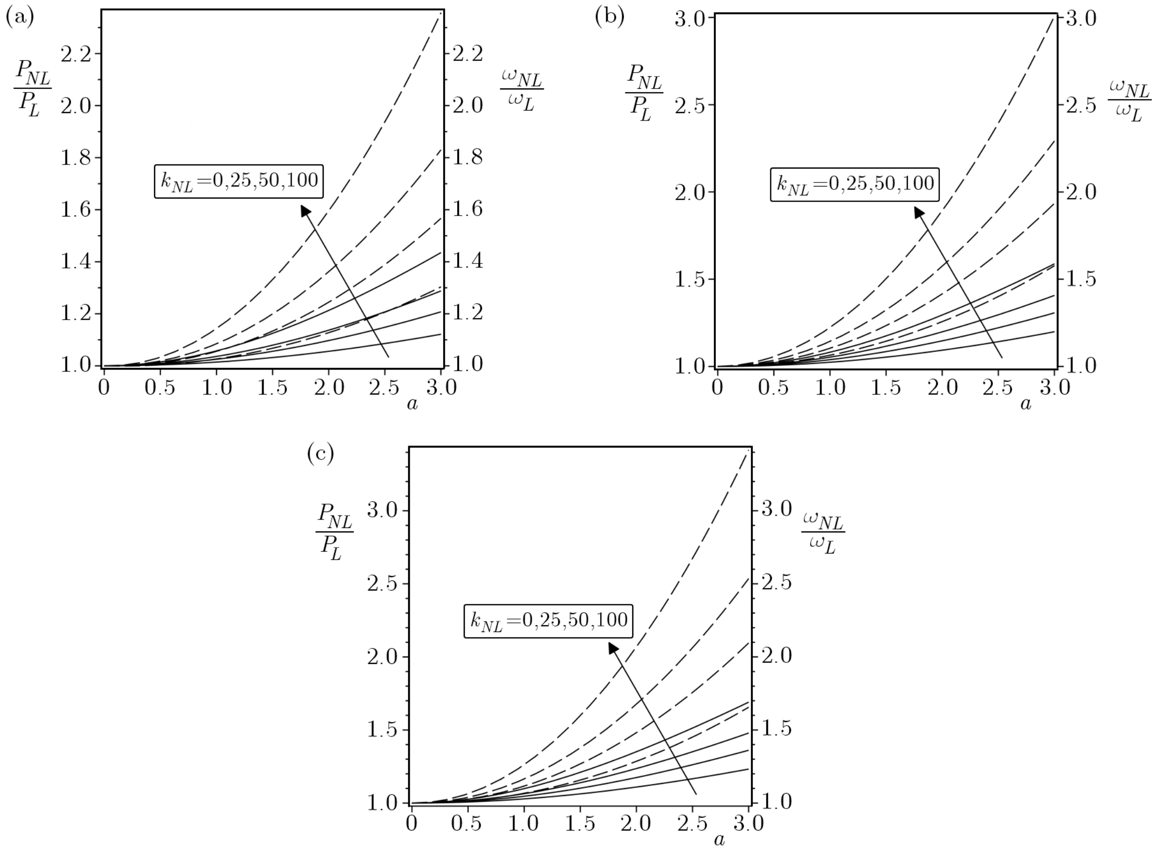

Fig. 5. Effect of the non-linear foundation stiffness on (a) SS FG, (b) CC FG and (c) CS FG beam for frequency solid line) and buckling load (dash line) ratios ($k_{NL} = 50$, $k_S = 50$, $P = 1$, $n = 2$)

Table 6. Effect of material inhomogenity on the frequency ratio of the FG beam ($k_L = 50$, $k_{NL} = 50$, $k_S = 5$)

	a	P	n					
			0	1	2	3	4	5
SS	0.5	0	1.02904918	1.02919972	1.02890989	1.02872178	1.02862044	1.02856965
		5	1.03859244	1.03879152	1.03840822	1.03815943	1.03802539	1.03795822
		10	1.05748102	1.05777493	1.05720904	1.05684167	1.05664372	1.05654453
CC	0.5	0	1.04011944	1.04035038	1.03990573	1.03961710	1.03946159	1.03938366
		5	1.04796154	1.04823660	1.04770699	1.04736319	1.04717794	1.04708511
		10	1.05961742	1.05995749	1.05930270	1.05887760	1.05864854	1.05853375
CS	0.5	0	1.04011944	1.04035038	1.03990573	1.03961710	1.03946159	1.03938366
		5	1.04796154	1.04823660	1.04770699	1.04736319	1.04717794	1.04708511
		10	1.05961742	1.05995749	1.05930270	1.05887760	1.05864854	1.05853375

6. Conclusion

Large amplitude vibration and post-buckling behavior of functionally graded beams rest on non-linear elastic foundation with simply supported, clamped-clamped and clamped-simply supported boundary conditions were investigated using the variational iteration method. This study is within the framework of Euler-Bernoulli's beam theory and von-Karman's type of the displacement-strain relationship. The accuracy of the method was investigated by comparing the results with those available from the literature and well-established by the Runge-Kutta

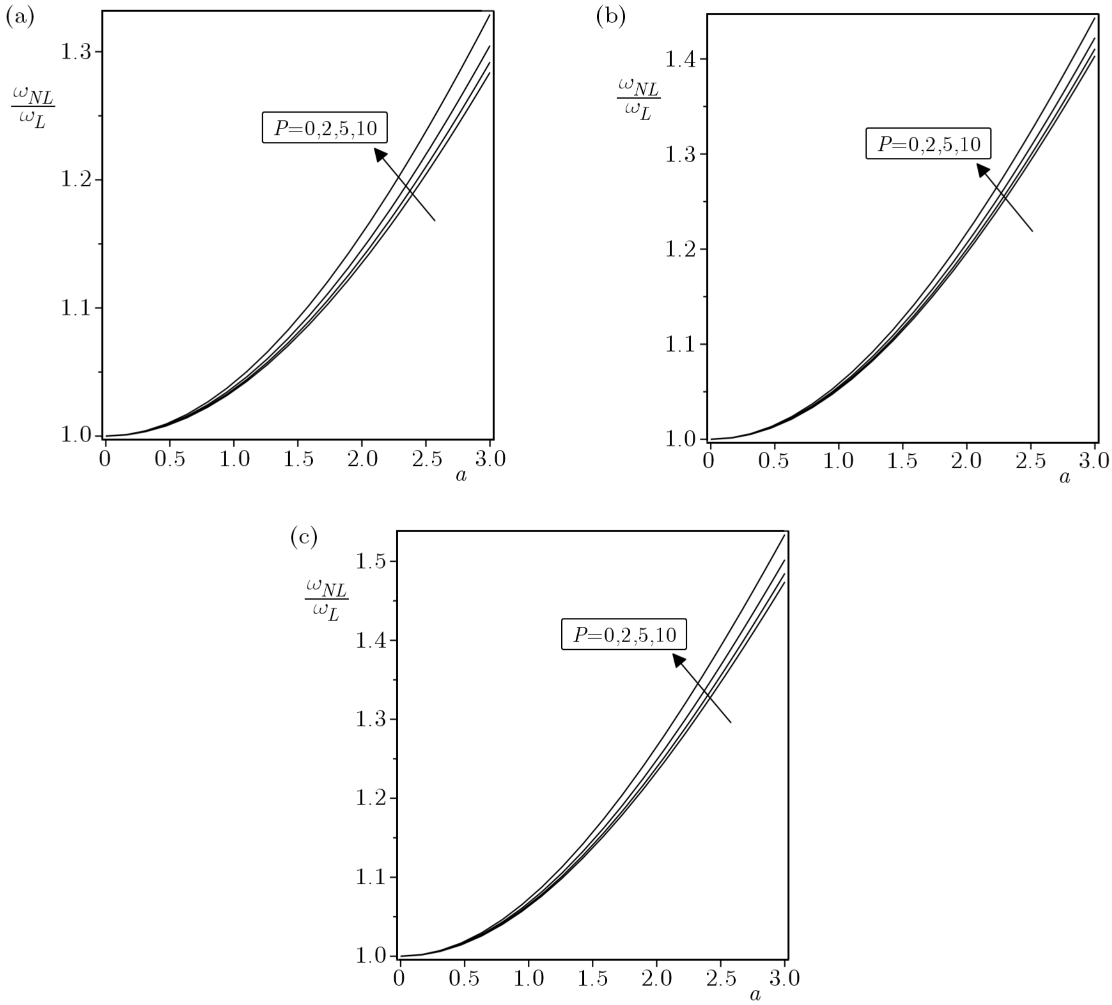

Fig. 6. Effect of the axial load on the frequency ratio of (a) SS FG, (b) CC FG and (c) CS FG beam ($k_{NL} = 50$, $k_S = 50$, $k_L = 50$, $n = 2$)

numerical method. The effects of foundation parameters, axial force, vibration amplitude, end supports and material inhomogeneity on non-linear dynamic behavior of FG beams were discussed in detail. As a result, the influence of linear and shear layers of the foundation is to weaken the non-linear behavior of the FG beam, whereas the effect of the non-linear foundation stiffness is to harden the beam response.

The presented expressions are convenient and efficient for the non-linear analysis of FG beams. Finally, it has been attempted to show the capabilities and wide-range applications of the VIM in solving such problems.

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Analityczne badania drgań o dużej amplitudzie i ugięcia po wyboczeniu belek z materiału gradientowego spoczywających na nieliniowo sprężystym podłożu

Streszczenie

W pracy przedstawiono analizę drgań nieliniowych i zjawisk następujących po wyboczeniu w belkach wykonanych z funkcjonalnych materiałów gradientowych (FGMs), spoczywających na nieliniowo sprężystym podłożu i jednocześnie poddanych osiowemu ściskaniu. Na podstawie teorii Eulera-Bernoulliego oraz przy uwzględnieniu geometrycznej nieliniowości von Karmana wyprowadzono cząstkowe równanie różniczkowe ruchu takich układów. Równanie to sprowadzono do postaci różniczkowej zwyczajnej za pomocą metody Galerkina. Na koniec, rozwiązano je analitycznie poprzez zastosowanie iteracyjnej metody wariacyjnej (VIM), a uzyskane rozwiązanie porównano z innymi, już istniejącymi i znanymi w literaturze, stwierdzając doskonałą zgodność. Otrzymano również nowe rezultaty w postaci określenia wpływu amplitudy drgań, sprężystości podłoża, wartości siły osiowej, rodzaju podparcia brzegów oraz niejednorodności materiału na częstości własne i obciążenie krytyczne belek gradientowych.

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