Optimal Control of Structural Dynamic Systems in One Space Dimension Using a Maximum Principle

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Abstract: A maximum principle is developed for a class of problems involving the optimal control of a system of linear hyperbolic equations in one space dimension that are not necessarily separable. An index of performance is formulated, which consists of vector functions of the state variable, their first- and second-order space derivatives and first-order time derivative, and a penalty function involving the open-loop control force vector. The solution of the optimal control problem can easily be shown to be unique using convexity arguments. The given maximum principle involves a Hamiltonian, which contains an adjoint vector function as well as an admissible control vector function. The maximum principle can be used to compute the optimal control vector function and is particularly suitable for problems involving the active control of structural elements for vibration suppression. A numerical example is given which studies the active control of a beam undergoing flexural and torsional vibrations. A comparison of the energies of controlled and uncontrolled beams indicates that the proposed control method is quite effective in damping out the vibrations of structural systems.

Key Words: Maximum principle for optimal control, distributed parameter systems, systems of hyperbolic equations

1. INTRODUCTION

The theory presented in this paper deals with the formulation and proof of a maximum principle associated with the optimal control of a system of hyperbolic equations with variable coefficients. These problems arise in the control of structural systems for vibration suppression. Structural control has been an active area of research for a number of years, and control is being used in many engineering applications ranging from aerospace to civil structures.

The seminal work pertaining to the theoretical considerations of the optimal control of distributed parameter systems is given by Lions (1971, 1972). Uses of maximum principles in optimal control problems include the studies given by Arutyunov et al. (1992), Basile and Mininni (1990), Chilton (1991), Fattorini (1985, 1993, 1994), Kaskosz (1990), Ledzewicz (1993), Sadigh-Esfandiari et al. (1990) and Sloss et al. (1989). In these studies maximum principles were derived for a number of cases and various qualitative aspects of control problems were investigated. Other work involving the optimal control of distributed parameter systems using a maximum principle includes the studies by Boltyansky and Poznyak (1999), De-Xing (1998), Kowalewski (1998), Margaliot and Langholz (1999), Sadek et al. (1998a, 1998b), and Wang and Chen (1999). An approach involving the use of dynamic programming was made by Belbas (1990). The formulation of the maximum principle with a view towards structural applications has been of limited nature. As these problems make up an important class in various fields of engineering, further study of the subject is needed.

A maximum principle which is easily applicable to structural dynamic problems was formulated for the optimal control of a hyperbolic equation in one space dimension by Sloss et al. (1995). An application of this maximum principle with numerical results is given by Bruch et al. (1995). However, a large class of structural vibration problems (such as beams and plates vibrating in more than one plane, frames, trusses, etc.) is governed by systems of hyperbolic equations. The maximum principle formulated in Sloss et al. (1995) is not useful for such systems. In dealing with systems, the mathematical formulation and derivation are more subtle and complicated. The present study is aimed at deriving a maximum principle which can be used to determine the optimal control solutions to damp out the vibrations of these structural systems. This is achieved by adapting a performance index which can represent the potential and kinetic energy of the system and penalizes the expenditure of control force. The uniqueness of the control function can easily be proved using convexity arguments. The maximum principle involves a Hamiltonian which relates an adjoint variable vector to the control vector. This approach leads to a coupled boundary, terminal and initial value problem, the solution of which can be obtained explicitly under certain conditions.

Although the maximum principle presented is restricted to equations of state involving at most four space derivatives, the methods used can be adapted without conceptual change to higher derivatives. These restrictions were introduced for clarity of presentation while preserving the applicability of the technique to problems arising in structural mechanics.

2. PHYSICAL PROBLEM FORMULATION

For structural systems, the equations of motion are a consequence of the extended Hamilton's principle. The motion of the system will be described by time t, $0 \leq t \leq t_f$, one space variable x, $0 \leq x \leq l$, and several dependent variables, where t_f is the terminal time. The principle is expressed in the form

$$\int_{t_1}^{t_2} \delta(KE - PE)\mathrm{d}t + \int_{t_1}^{t_2} \overline{\delta W}_{nc}\mathrm{d}t = 0, \tag{1}$$

where *KE* is the kinetic energy, *PE* is the potential energy of the system, $\overline{\delta W}_{nc}$ is the virtual work due to the non-conservative distributed forces, and t_1 and t_2 are times at which the state is known, i.e.

$$\delta W = 0 \quad \text{at} \quad t = t_1, t_2.$$

The state variable row-vector is given by

$$W = W(x,t) = [W_1, W_2, ..., W_N]^{\mathrm{T}}, \quad 0 \le x \le l$$

with $W_j(x,t)$, $1 \le j \le N$, denoting the N dependent variables and superscript "T" denotes the transpose.

The kinetic energy KE will be assumed to be of the form

$$KE = \frac{1}{2}\int_0^l W_t^{\mathrm{T}} M(x) W_t \mathrm{d}x, \tag{2}$$

where $M(x)$ is an $N \times N$ symmetric positive definite matrix for all $x, 0 \le x \le l$, and the subscript t refers to differentiation with respect to t. It follows that

$$\int_{t_1}^{t_2} \delta KE \mathrm{d}t = -\int_{t_1}^{t_2}\int_0^l \delta W^{\mathrm{T}} M(x) W_{tt} \mathrm{d}x\mathrm{d}t. \tag{3}$$

The potential energy PE will be assumed to be of the form

$$\begin{aligned} PE(W) &= \frac{1}{2}\int_0^l W_{xx}^{\mathrm{T}} P_2(x) W_{xx} \mathrm{d}x + \frac{1}{2}\int_0^l W_x^{\mathrm{T}} P_1(x) W_x \mathrm{d}x \\ &+ \frac{1}{2}\int_0^l \left(\overline{A}W_x - BW\right)^{\mathrm{T}} P_0(x) \left(\overline{A}W_x - BW\right) \mathrm{d}x, \end{aligned} \tag{4}$$

in which $P_k(x)$, $(k = 0, 1, 2)$, are $N \times N$ symmetric matrices for all $x, 0 \le x \le l$, each component of which is in $L^\infty(0,l)$, and $\overline{A}$ and B are $N \times N$ matrices of constants, not necessarily non-singular and the x subscripts refer to differentiation with respect to x. It is assumed for some $\lambda > 0$ and $\alpha > 0$

$$PE(W) + \lambda\,|W|_{L^2(0,l)} \ge \alpha\,\|W\|_{H^2(0,l)}, \quad |U|_{L^2(0,l)} = \int_0^l U^{\mathrm{T}}(x)\,U(x)\,\mathrm{d}x \tag{5}$$

when P_2 is positive definite and if $P_2 \equiv 0$, then $H^2(0,l)$ is replaced by $H^1(0,l)$. It follows that

$$\begin{aligned} \int_{t_1}^{t_2} \delta PE \mathrm{d}t &= \int_{t_1}^{t_2} \Bigg\{ \int_0^l \delta W^{\mathrm{T}} \Big[(P_2(x)W_{xx})_{xx} - (P_1(x)W_x)_x - B^{\mathrm{T}} P_0(x)\left(\overline{A}W_x - BW\right) \\ &- \overline{A}^{\mathrm{T}} \left(P_0(x)\left(\overline{A}W_x - BW\right)\right)_x \Big] \mathrm{d}x + \Big[\delta W_x^{\mathrm{T}} P_2(x) W_{xx} - \delta W^{\mathrm{T}} \big((P_2(x)W_{xx})_x \\ &+ P_1(x)W_x - \overline{A}^{\mathrm{T}} P_0(x)\left(\overline{A}W_x - BW\right)\big) \Big]_{x=0}^{x=l} \Bigg\} \mathrm{d}t. \end{aligned} \tag{6}$$

The virtual work due to the non-conservative distributed forces is given by

$$\overline{\delta W}_{nc}(t) = \int_0^l \delta W^{\mathrm{T}}[F - C(x)W_t]\mathrm{d}t, \tag{7}$$

where F is an $N \times 1$ column vector of externally applied control forces, i.e. $F = F(x,t) = [F_1(x,t), F_2(x,t), ..., F_N(x,t)]^{\mathrm{T}}$, $F_i(x,t) \in L^2(0, t_f, L^2(0,l))$ and $C(x)$ is the $N \times N$ viscous coefficient matrix (symmetric positive semidefinite matrix for all x, $0 \le x \le l$).

Combining equations (1), (3), (6), and (7) leads to the following equations of state

$$\begin{aligned} M(x)W_{tt} &+ C(x)W_t + (P_2(x)W_{xx})_{xx} - (P_1(x)W_x)_x - \overline{A}^{\mathrm{T}}\left[P_0(x)\left(\overline{A}W_x - BW\right)\right]_x \\ &- B^{\mathrm{T}}P_0(x)\left(\overline{A}W_x - BW\right) = F(x,t), \ 0 < x < l, \ 0 < t < t_f. \end{aligned} \tag{8}$$

Combining equations (1) and (6) yields the following proper conditions when appropriate component parts are taken equal to zero at $x = 0$ and $x = l$, i.e.

$$\delta W_x^{\mathrm{T}}[P_2(x)W_{xx}] = 0 \tag{9}$$

and

$$\delta W^{\mathrm{T}}\left\{\overline{A}^{\mathrm{T}}P_0(x)BW - \left[\overline{A}^{\mathrm{T}}P_0(x)\overline{A} - P_1(x)\right]W_x + (P_2(x)W_{xx})_x\right\} = 0. \tag{10}$$

As an example, consider a Timoshenko beam without damping which is derived variationally from (Magrab, 1979)

$$I = \frac{1}{2}\int_0^l \left[k_b^2 EI\psi_x^2 + k_s^2 AG(W_x - \psi)^2 - \rho A W_t^2 - \rho A r^2 \psi_t^2 - 2FW - 2M_a\psi\right]\mathrm{d}x.$$

Comparing, it is seen that

$$W(x,t) = [w(x,t), \psi(x,t)]^{\mathrm{T}}, \qquad M(x) = \begin{bmatrix} -\rho A & 0 \\ 0 & -\rho A r^2 \end{bmatrix},$$

$$C(x) = P_2(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad P_1(x) = -\begin{bmatrix} 0 & 0 \\ 0 & k_b^2 EI \end{bmatrix},$$

$$\overline{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad P_0(x) = -k_s^2 AG\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$F(x,t) = -[F_a(x,t), M_a(x,t)]^{\mathrm{T}}$$

from which the following equations of motion are obtained

$$\begin{aligned} \left[k_s^2 AG(w_x - \psi)\right]_x - \rho A w_{tt} &= -F_a(x,t), \\ \left(k_b^2 EI\psi_x\right)_x + k_s^2 AG(w_x - \psi) - \rho A r^2 \psi_{tt} &= M_a(x,t) \end{aligned}$$

with boundary conditions at $x = 0, l$

$$w\left[k_s^2 AG(w_x - \psi)\right] = 0$$

and

$$\psi\left(k_b^2 EI\psi_x\right) = 0.$$

Examples of these are:
clamped end

$$w = 0 \quad \text{and} \quad \psi = 0;$$

simply supported (hinged) end

$$w = 0 \quad \text{and} \quad EI\psi_x = 0;$$

free end

$$k_b^2 EI\psi = 0 \quad \text{and} \quad k_s^2 AG(w_x - \psi) = 0.$$

Here, $w(x, t)$ is the transverse displacement, $\psi(x, t)$ is the angle of rotation of the cross-section due to bending only, A is the cross-sectional area of the beam, ρ is the beam density, $r = \sqrt{I/A}$ is the radius of gyration of the beam cross-section, k_s and k_b are functions of the shape of the beam cross-section and the mode of vibration, E is the Young's modulus, I is the moment of inertia of the beam cross-section, G is the shear modulus of the beam material, $F_a(x, t)$ is the applied control force per unit length, and $M_a(x, t)$ is the applied control moment per unit length.

For this example, we introduce the new dependent variable

$$v(x, t) = W(x, t) - \int_0^x \psi(s, t)\mathrm{d}s$$

with corresponding boundary conditions:
clamped

$$\left\{\begin{array}{l} v(0, t) = 0 \\ \psi(0, t) = 0 \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{l} v(l, t) = -\int_0^l \psi(s, t)\mathrm{d}s \\ \psi(l, t) = 0 \end{array}\right\};$$

simply supported

$$\left\{\begin{array}{l} v(0, t) = 0 \\ EI\psi_x(0, t) = 0 \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{l} v(l, t) = -\int_0^l \psi(s, t)\mathrm{d}s \\ EI\psi_x(l, t) = 0 \end{array}\right\};$$

free

$$\left\{ \begin{array}{l} k_s^2 AG v_x(0,t) = 0 \\ k_b^2 EI \psi(0,t) = 0 \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{l} k_s^2 AG v_x(l,t) = 0 \\ k_b^2 EI \psi(l,t) = 0 \end{array} \right\}.$$

Noting that $P_2(x) \equiv 0$ and setting $U = [v, \psi]^{\mathrm{T}}$, the potential energy satisfies

$$PE[U] = \frac{1}{2}\int_0^l \left[k_b^2 EI \psi_x^2 + k_s^2 AG v_x^2\right] \mathrm{d}x \geq m_0 \int_0^l \left(\psi_x^2 + v_x^2\right) \mathrm{d}x,$$

where

$$m_0 = \min\left\{\frac{1}{2}k_b^2 EI, \frac{1}{2}k_s^2 AG\right\}$$

and condition (5) is satisfied.

3. CONTROL PROBLEM FORMULATION

Let

$$\Gamma_1(x,t) = P_2(x) W_{xx}$$

and

$$\Gamma_2(x,t) = \overline{A}^{\mathrm{T}} P_0(x) B W - \left[\overline{A}^{\mathrm{T}} P_0(x) \overline{A} + P_1(x)\right] W_x + (P_2(x) W_{xx})_x.$$

In general, the boundary conditions in equations (9) and (10) can be written in component form at $x = 0$ and $x = l$ as

$$W_{jx}(x,t) = 0 \quad \text{or} \quad \Gamma_{1j}(x,t) = 0 \tag{11}$$

and

$$W_j(x,t) = 0 \quad \text{or} \quad \Gamma_{2j}(x,t) = 0 \tag{12}$$

for $j = 1, 2, ..., N$.

Let $\overline{P}_1(x) = -P_1(x)$ and

$$< U, V > = \int_0^l \left[U_{xx}^{\mathrm{T}} P_2(x) V_{xx} + U_x^{\mathrm{T}} \overline{P}_1(x) V_x + \left(\overline{A} U_x - BU\right)^{\mathrm{T}} P_0(x) \left(\overline{A} V_x - BV\right)\right] \mathrm{d}x$$

then $< U, V > = < V, U >$. If in addition to $U, V \in H^2(0,l)$, or $H^1(0,l)$ if $P_2 \equiv 0$, it is assumed that the *j*th component of $\langle U, V \rangle$ satisfies equations (11) and (12), then

$$< U, V > = 0.$$

Let U_{ad} be the set of admissible controls defined by

$$U_{ad} = \left\{G(x,t) | G(x,t) = [G_1(x,t), G_2(x,t), ..., G_N(x,t)]^{\mathrm{T}}, G_j(x,t) \in L^2(L^2(0,l); 0, t_f)\right\}.$$

Consider equation (8) in which $F(x,t) \in U_{ad}$ and the initial conditions for the problem are

$$W(x,0;F) = \mathbf{\Phi}(x) \quad , \quad W_t(x,0;F) = \mathbf{\Psi}(x) \tag{13}$$

with

$$\mathbf{\Phi} = \left[\phi_1, ..., \phi_N\right]^{\mathrm{T}} \quad , \quad \mathbf{\Psi} = \left[\psi_1, ..., \psi_N\right]^{\mathrm{T}}$$

in which $\phi_i \in H^2(0,l)$ and $\psi_i \in L^2(0,l)$, $i = 1, 2, ..., N$ (or $\phi_i \in H^1(0,l)$ when $P_2 \equiv 0$).

Consider the index of performance

$$\begin{aligned} J[F] &= \frac{1}{2}\int_0^l [W^{\mathrm{T}}(x,t_f)g_0(x)W(x,t_f) + W_x^{\mathrm{T}}(x,t_f)g_1(x)W_x(x,t_f) \\ &+ W_{xx}^{\mathrm{T}}(x,t_f)g_2(x)W_{xx}(x,t_f) + W_t^{\mathrm{T}}(x,t_f)g_3(x)W_t(x,t_f)]\mathrm{d}x \\ &+ \int_0^{t_f}\int_0^l F^{\mathrm{T}}(x,t)g_4(x)F(x,t)\mathrm{d}x\mathrm{d}t \end{aligned} \tag{14}$$

in which $g_j(x)$, $j = 1, 2, 3, 4$ are $N \times N$ constant symmetric positive definite matrices.

For $F \in U_{ad}$ and initial conditions of the assumed form, the equation of state (8) admits a unique solution $W(x,t;F)$ (see Lions, 1971) with

$$\begin{aligned} W_j(x,t;F) &\in L^2\left(0, t_f; H^2(0,l)\right) \quad \text{and is continuous on } [0, t_f] \\ W_{jt}(x,t;F) &\in L^2\left(0, t_f; L^2(0,l)\right) \quad \text{and is continuous on } [0, t_f] \end{aligned}$$

for $j = 1, 2, \ldots, N$, and hence it is plausible to use the index of performance introduced.

The optimal control problem can now be stated, as follows.

Determine an optimal control vector $F^* \in U_{ad}$ so that $J[F]$ is minimized, i.e.

$$J[F^*] \leq J[F] \quad \text{for all} \quad F \in U_{ad}.$$

We wish to consider the situation when the state variable satisfies the boundary conditions (11) and (12).

The proof of uniqueness of the optimal control follows easily from the convexity of the index of performance. Assuming that an optimal control exists, our main objective will be in deriving a maximum principle that can be used to determine the optimal control. To this end we shall introduce a variable $V(x,t)$ adjoint to the state variable $W(x,t)$ and show that the optimal control function is related to this adjoint variable. To achieve this, a Hamiltonian functional will be introduced involving the adjoint variable corresponding to the optimal control and the optimal control. It will be shown that the maximum value of this functional over all admissible controls is achieved for that unique control that is the optimal control function. Due to the nature of the Hamiltonian, the relation between the optimal control and the adjoint variable can be seen to be an explicit expression of the optimal control in terms of the adjoint variable, and consequently it is useful in finding the solution of the optimal control problem.

4. MAXIMUM PRINCIPLE STATEMENT

In this section, we state the maximum principle for a system of hyperbolic equations in one space dimension. For $F \in U_{ad}$, let $W(x,t) = W(x,t;F)$ satisfy equation (8) with initial conditions (13) and boundary conditions (11) and (12). For $F^0 \in U_{ad}$ let $W^0(x,t) = W(x,t;F^0)$ and let $V^0(x,t) = V(x,t;F^0)$ be the corresponding adjoint variable, i.e. $V^0(x,t)$ satisfies equation (8) with $F = 0$ and with C replaced by $-C$ and boundary conditions (11) and (12) along with the terminal conditions

$$V_t^{0^{\mathrm{T}}} M - \left[V^{0^{\mathrm{T}}}\right] C = W^{0^{\mathrm{T}}} g_o - W_x^{0^{\mathrm{T}}} g_1 + W_{xx}^{0^{\mathrm{T}}} g_2 \quad \text{at} \quad t = t_f \tag{15}$$

$$-V^{0^{\mathrm{T}}} M = W_t^{0^{\mathrm{T}}} g_3 \quad \text{at} \quad t = t_f. \tag{16}$$

Let

$$H(x,t;V,F) = V^{\mathrm{T}} F - F^{\mathrm{T}} g_4 F \quad \text{(Hamiltonian)} \tag{17}$$

and

$$S(x,t_f;W) = \left[W_x^{\mathrm{T}} g_1 - \left(W_{xx}^{\mathrm{T}} g_2\right)_x\right] \Delta W + \left(W_{xx}^{\mathrm{T}} g_2\right) \Delta W_x \tag{18}$$

in which $\Delta W = W(x,t;F) - W(x,t;F^0)$ is a solution of the equation of state subject to homogeneous initial conditions. The maximum principle states that if

(i) only boundary conditions compatible with

$$S(x,t_f;W)\Big|_{x=0}^{x=l} = 0 \tag{19}$$

are considered and if

(ii)
$$H\left[x,t;V^0,F^0\right] = \max_{F \in U_{ad}} H\left[x,t;V^0,F\right]. \tag{20}$$

then

$$J(F^0) \le J(F) \quad \text{for all} \quad F \in U_{ad}.$$

That is, F^0 is an optimal control and due to uniqueness, the optimal control $F^*(x,t)$ is given by $F^0(x,t)$. The maximum principle then provides a method for determining the optimal control vector $F^*(x,t)$ assuming that a solution of the optimal control problem exists.

5. PROOF OF THE MAXIMUM PRINCIPLE

Setting

$$\begin{aligned} L^4(x) &= 2P_2(x), L^3(x) = 2(P_2(x))_x, L^2(x) = -P_1(x) + (P_2(x))_{xx} - \overline{A}^{\mathrm{T}} P_0(x)\overline{A}, \\ L^1(x) &= -(P_1(x))_x - \overline{A}^{\mathrm{T}}(P_0(x))_x\overline{A} + \overline{A}^{\mathrm{T}} P_0(x)B - B^{\mathrm{T}} P_0\overline{A}, \quad \text{and} \\ L^0(x) &= \overline{A}^{\mathrm{T}}(P_0(x))_x B + B^{\mathrm{T}} P_0(x)B. \end{aligned}$$

(For the case when $P_1(x)$ is a compressive force, $P_1(x) < 0$.) It follows that

$$\overline{L}[W] = F(x,t), \tag{21}$$

where

$$\overline{L}[W] = M(x)W_{tt} + C(x)W_t + L(x)[W]$$

with

$$L[W] = L^0(x)W + L^1(x)W_x + L^2(x)W_{xx} + L^3(x)W_{xxx} + L^4(x)W_{xxxx}.$$

Let

$$\tilde{L}[W] = \overline{L}[W] - F, \qquad \tilde{L}\left[W^0\right] = \overline{L}\left[W^0\right] - F^0$$

with W and W^0 satisfying the initial conditions (13) and boundary conditions (11) and (12). If

$$\triangle W = W - W^0, \quad \triangle F = F - F^0$$

in which

$$W = W(x,t;F), \quad W^0 = W(x,t;F^0),$$

then

$$\tilde{L}[\triangle W] = \bar{L}[\triangle W] - \triangle F = 0$$

and the initial conditions become

$$\triangle W(x,0;F) = 0, \quad \triangle W_t(x,0;F) = 0. \tag{22}$$

Let

$$\beta[V^0, \Delta W] = V_x^{0^{\mathrm{T}}}[P_2\Delta W_{xx}] + V^{0^{\mathrm{T}}}\left\{\overline{A}^{\mathrm{T}} P_0 B\Delta W + \left[-\overline{A}^{\mathrm{T}} P_0\overline{A} + P_1\right]\Delta W_x + (P_2\Delta W_{xx})_x\right\}$$

and note that

$$\beta\left[V^0, \Delta W\right]_{x=0}^{x=l} = 0. \tag{23}$$

Consider

$$\begin{aligned}
0 &= \int_0^{t_f}\int_0^l \left\{V^{0^{\mathrm{T}}}\overline{L}[\Delta W] - \left(L^*\left[V^0\right]\right)^{\mathrm{T}}\Delta W - V^{0^{\mathrm{T}}}\Delta F\right\}\mathrm{d}x\mathrm{d}t \\
&= \int_0^{t_f}\int_0^l \left\{V^{0^{\mathrm{T}}}[M\Delta W_{tt} + C\Delta W_t] + V^{0^{\mathrm{T}}}L[\Delta W]\right. \\
&- \left.\left[\left(MV_{tt}^0\right)^{\mathrm{T}} - \left(CV_t^0\right)^{\mathrm{T}} + \left(L\left[V^0\right]\right)^{\mathrm{T}}\right]\Delta W - V^{0^{\mathrm{T}}}\Delta F\right\}\mathrm{d}x\mathrm{d}t \\
&= \int_0^{t_f}\int_0^l \left\{V^{0^{\mathrm{T}}}[M\Delta W_{tt} + C\Delta W_t] - \left[\left(MV_{tt}^0\right)^{\mathrm{T}} - \left(CV_t^0\right)^{\mathrm{T}}\right]\Delta W - V^{0^{\mathrm{T}}}\Delta F\right\}\mathrm{d}x\mathrm{d}t
\end{aligned}$$

since

$$\int_0^l V^{0^{\mathrm{T}}}L[\Delta W]\mathrm{d}x = \int_0^l L\left[V^{0^{\mathrm{T}}}\right]\Delta W\mathrm{d}x + \beta\left[V^0, \Delta W\right]_{x=0}^{x=l} = \int_0^l L\left[V^{0^{\mathrm{T}}}\right]\Delta W\mathrm{d}x. \tag{24}$$

Hence,

$$\begin{aligned}
0 &= \int_0^l\int_0^{t_f}\frac{\partial}{\partial t}\left\{V^{0^{\mathrm{T}}}M\Delta W_t - V_t^{0^{\mathrm{T}}}M\Delta W\right\}\mathrm{d}t\mathrm{d}x + \frac{\partial}{\partial t}\left\{V^{0^{\mathrm{T}}}C\Delta W\right\}\mathrm{d}t\mathrm{d}x \\
&- \int_0^{t_f}\int_0^l V^{0^{\mathrm{T}}}\Delta F\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Integrating the first and second integrals with respect to time and taking into consideration equation (22) yields

$$0 = \int_0^l \left\{V^{0^{\mathrm{T}}}M\Delta W_t - \left(V_t^{0^{\mathrm{T}}}M - V^{0^{\mathrm{T}}}C\right)\Delta W\right\}_{t=t_f}\mathrm{d}x - \int_0^{t_f}\int_0^l V^{0^{\mathrm{T}}}\Delta F\mathrm{d}x\mathrm{d}t.$$

Making use of the terminal conditions in equations (15) and (16) gives

$$\begin{aligned}
0 &= -\int_0^l \left\{W_t^{0^{\mathrm{T}}}g_3\Delta W_t + \left[W^{0^{\mathrm{T}}}g_0 - \left(W_x^{0^{\mathrm{T}}}g_1\right)_x + \left(W_{xx}^{0^{\mathrm{T}}}g_2\right)_{xx}\right]\Delta W\right\}_{t=t_f}\mathrm{d}x \\
&- \int_0^{t_f}\int_0^l V^{0^{\mathrm{T}}}\Delta F\mathrm{d}x\mathrm{d}t \\
0 &= -\int_0^l \left\{W_t^{0^{\mathrm{T}}}g_3\Delta W_t + W^{0^{\mathrm{T}}}g_0\Delta W + W_x^{0^{\mathrm{T}}}g_1\Delta W_x + W_{xx}^{0^{\mathrm{T}}}g_2\Delta W_{xx}\right\}_{t=t_f}\mathrm{d}x \\
&+ S(x, t_f)\Big|_{x=0}^{x=l} - \int_0^{t_f}\int_0^l V^{0^{\mathrm{T}}}\Delta F\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where use has been made of integration by parts and the definition of *S(x,t).* Note that

$$\begin{aligned} W_t^{\mathrm{T}} g_3 W_t &= W_t^{0^{\mathrm{T}}} g_3 W_t^0 + 2\left(W_t^{\mathrm{T}} - W_t^{0^{\mathrm{T}}}\right) g_3 W_t^0 \\ &+ \left(W_t^{\mathrm{T}} - W_t^{0^{\mathrm{T}}}\right) g_3 \left(W_t - W_t^0\right) \geq W_t^0 g_3 W_t^0 + 2\Delta W_t^{\mathrm{T}} g_3 W_t^0 \end{aligned} \tag{25}$$

since g_3 is positive definite. Hence,

$$W_t^{0^{\mathrm{T}}} g_3 W_t \leq \frac{1}{2}\left(W_t^{\mathrm{T}} g_3 W_t + W_t^{0^{\mathrm{T}}} g_3 W_t^0\right).$$

Similar inequalities hold for the second, third, and fourth terms in equation (25).

Hence,

$$\begin{aligned} 0 \geq & -\int_0^l \frac{1}{2}\left\{W_t^{\mathrm{T}} g_3 W_t + W^{\mathrm{T}} g_0 W + W_x^{\mathrm{T}} g_1 W_x + W_{xx}^{\mathrm{T}} g_2 W_{xx}\right. \\ & \left. - W_t^{0^{\mathrm{T}}} g_3 W_t^0 - W^{0^{\mathrm{T}}} g_0 W^0 - W_x^{0^{\mathrm{T}}} g_1 W_x^0 - W_{xx}^{0^{\mathrm{T}}} g_2 W_{xx}^0\right\}_{t=t_f} \mathrm{d}x \\ & -\int_0^{t_f} \int_0^l V^{0^{\mathrm{T}}}\left(F - F^0\right) \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Thus

$$\begin{aligned} 0 \geq & -J(F) + J(F^0) + \int_0^{t_f} \int_0^l F^{\mathrm{T}} g_4 F \mathrm{d}x\mathrm{d}t - \int_0^{t_f} \int_0^l F^{0^{\mathrm{T}}} g_4 F^0 \mathrm{d}x\mathrm{d}t \\ & -\int_0^{t_f} \int_0^l V^{0^{\mathrm{T}}} F \mathrm{d}x\mathrm{d}t + \int_0^{t_f} \int_0^l V^{0^{\mathrm{T}}} F^0 \mathrm{d}x\mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} J(F) - J(F^0) &\geq \int_0^{t_f} \int_0^l \left\{V^{0^{\mathrm{T}}} F^0 - F^{0^{\mathrm{T}}} g_4 F^0 - V^{0^{\mathrm{T}}} F + F^{\mathrm{T}} g_4 F\right\} \mathrm{d}x\mathrm{d}t \\ &= \int_0^{t_f} \int_0^l \left\{V^{0^{\mathrm{T}}} F^0 - F^{0^{\mathrm{T}}} g_4 F^0 - V^{0^{\mathrm{T}}} F + F^{\mathrm{T}} g_4 F\right\} \mathrm{d}x\mathrm{d}t \geq 0. \end{aligned}$$

Since by uniqueness there is at most one optimal control F^*, it follows that

$$F^0 = F^*,$$

and the proof of the maximum principle is complete.

6. NUMERICAL EXAMPLE

Consider the equations of motion of a beam undergoing flexural and torsional vibrations with no damping on the region $0 \leq x \leq l$, $0 \leq t \leq t_f$

$$\begin{aligned} \rho A\ddot{w}_1 - \rho Ac\ddot{w}_2 + K_1 w_1 + EI_z w_{1xxxx} &= F_1(x,t) \\ -\rho Ac\ddot{w}_1 + \rho(Ac^2 + I_p)\ddot{w}_2 + K_2 w_2 - Rw_{2xx} &= F_2(x,t). \end{aligned} \tag{26}$$

Here, w_1 is the deflection function, w_2 is the angle of twist, ρ is the mass per unit volume, A is the area, c is the distance of the shear center axis (x-axis) from the centroid, I_p is the centroid polar moment of inertia of the cross-section, R is the torsional rigidity, EI_z is the flexural rigidity with respect to the z-axis, F_1 is the distributed control force, F_2 is the distributed control torque, and K_1, K_2 are the elastic restoring constants, with simply supported boundary conditions

$$\begin{aligned} w_j(0,t) &= 0, \quad w_j(l,t) = 0, \quad j = 1,2 \\ w_{1xx}(0,t) &= 0, \quad w_{1xx}(l,t) = 0. \end{aligned} \tag{27}$$

Note that with $W = [w_1, w_2]^{\mathrm{T}}$

$$\begin{aligned} \Gamma_1(x,t;W) &= P_2 W_{xx}(x,t) = \begin{bmatrix} EI_z w_1(x,t) \\ 0 \end{bmatrix}, \\ \Gamma_2(x,t;W) &= P_1 W_x(x,t) + P_2 W_{xxx}(x,t) = \begin{bmatrix} EI_z w_{1xxx}(x,t) \\ -Rw_{2x}(x,t) \end{bmatrix}. \end{aligned}$$

and it is clear that the simply supported case is a candidate for the boundary conditions treated.

The initial conditions are taken to be

$$\begin{aligned} [w_1(x,0), w_2(x,0)] &= [0,0] \\ [w_{1t}(x,0), w_{2t}(x,0)] &= \left[\sin(\pi x/l), \sin(\pi x/l)\right]. \end{aligned} \tag{28}$$

The eigenfunctions of the system (26) are

$$\varphi_n(x) = \sin n\pi x/l, \qquad \psi_n(x) = \sin n\pi x/l. \tag{29}$$

In the notation of the paper

$$M = \begin{bmatrix} \rho A & -\rho Ac \\ -\rho Ac & \rho(Ac^2 + I_p) \end{bmatrix}$$

where

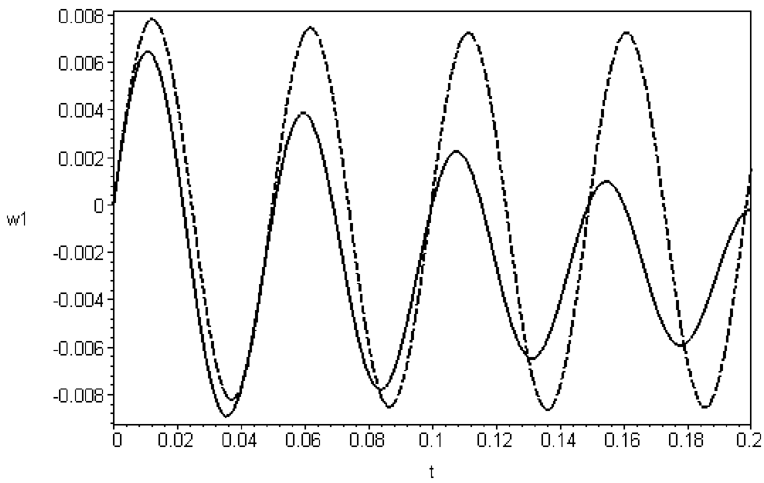

Figure 1. Controlled (solid line) and uncontrolled (dashed line) displacement.

$$
\begin{aligned}
P_0 &= \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & 0 \\ 0 & -R \end{bmatrix}, \quad P_2 = \begin{bmatrix} EI_z & 0 \\ 0 & 0 \end{bmatrix}, \\
\bar{A} &= 0^{2\times 2}, \quad B = I^{2\times 2}, \quad C = 0^{2\times 2}.
\end{aligned}
$$

The elastic properties of the beam are specified as $l = 20.0$ ft, $c = 1.0$ ft, $\rho A = 4.65$ lb ft^{-2} s^2, $EI_z = 1.00 \times 10^6$ lb ft ^{2}, $\rho I_p = 15.34$ lb s^2, $R = 1.00 \times 10^7$ lb ft^2, and $K_1 = K_2 = 0$. The weighting matrices in equation (14) are chosen as $g_0 = I$, $g_1 = 0$, $g_2 = g_3 = I$ and $g_4 = \begin{bmatrix} 10^{-5} & 0 \\ 0 & 10^{-5} \end{bmatrix}$. In this case, the first integral of the objective function (14) is given by

$$
E\,[w_1, w_2] = \frac{1}{2}\int_0^l \left(w_1^2 + w_2^2 + w_{1xx}^2 + w_{2xx}^2 + w_{1t}^2 + w_{2t}^2\right) \mathrm{d}x. \tag{30}
$$

The terminal time is taken as $t_f = 0.2$ s. First, the behaviors of the deflection w_1 and angle of twist w_2 are investigated. Figures 1 and 2 show the curves of $w_1(l/2, t)$ and $w_2(l/2, t)$ plotted against time $t \in [0, t_f]$ for the controlled and uncontrolled beams. It is observed that the controlled quantities decrease steadily as time increases, even though at $t = t_f$ the differences between the controlled and the uncontrolled quantities are minor.

Figures 3 and 4 show the time derivatives of the above quantities, i.e. $w_{1t}(l/2, t)$ and $w_{2t}(l/2, t)$, respectively, plotted against time $t \in [0, t_f]$. It is observed that at the final time the velocities are much smaller than the corresponding uncontrolled quantities. It is noted that the performance index includes functions of w_1 and w_2 and their time derivatives. Magnitudes of these functions and their velocities at the final time can be adjusted by selecting the weighting factors g_i accordingly.

The curves of $E(w_1, w_2)$ given by equation (30), which provides a measure of the energy of the vibrating beam, are shown in Figure 5 for controlled and uncontrolled beams.

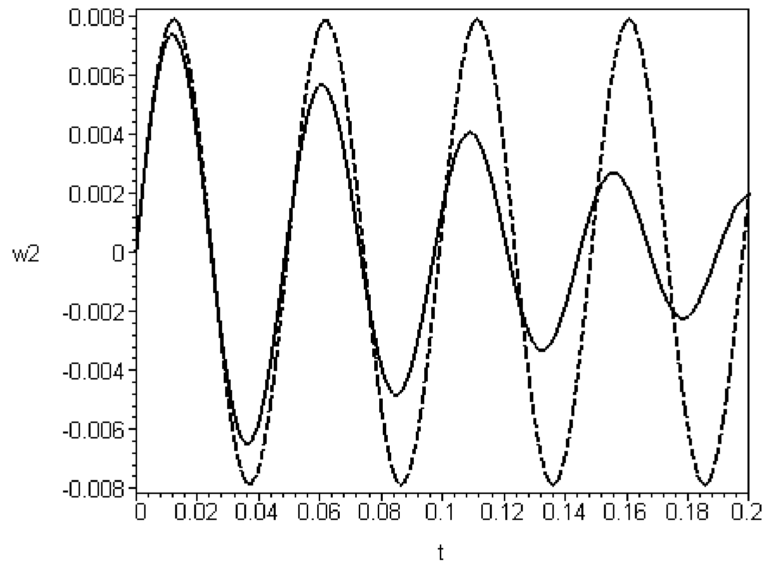

Figure 2. Controlled (solid line) and uncontrolled (dashed line) angle of twist.

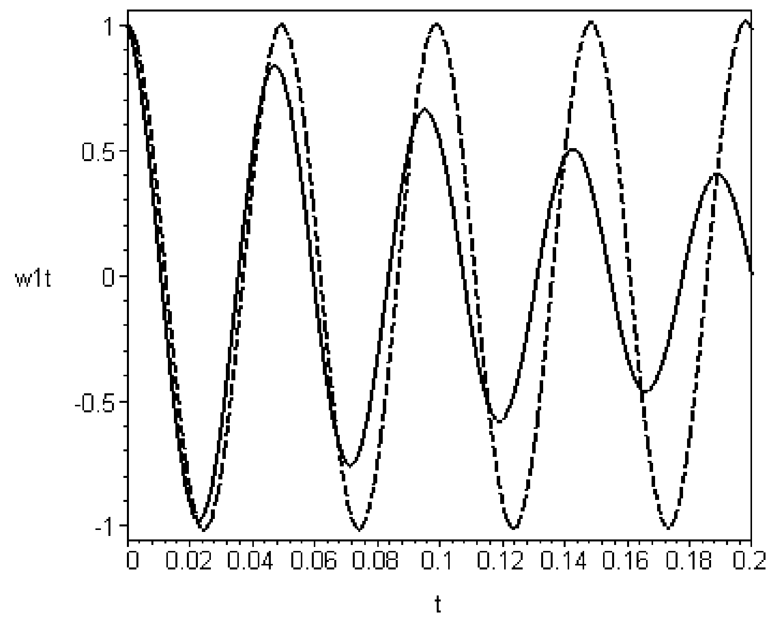

Figure 3. Controlled (solid line) and uncontrolled (dashed line) velocity.

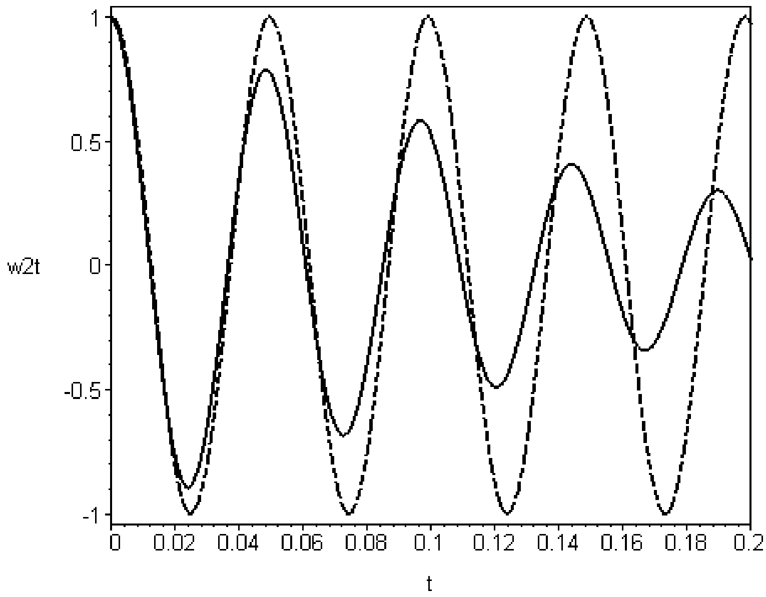

Figure 4. Controlled (solid line) and uncontrolled (dashed line) angular velocity.

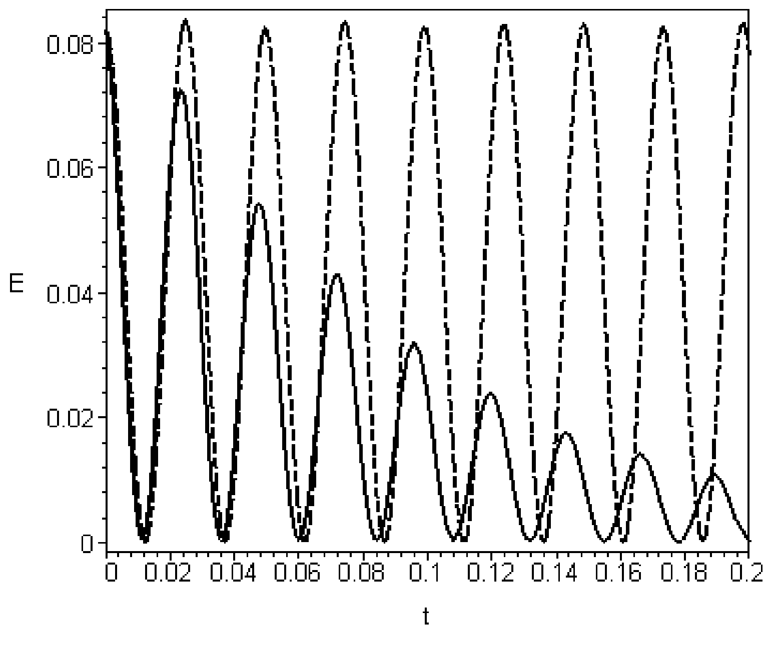

Figure 5. Controlled (solid line) and uncontrolled (dashed line) energy.

The controlled energy decreases as a damped sinusoidal function as time increases and is much smaller at $t = t_f$ compared to the uncontrolled beam. The substantial decrease in the energy with increasing time gives an indication of the effectiveness of the proposed control algorithm.

7. CONCLUSIONS

A maximum principle has been given for the optimal control of distributed parameter systems governed by a system of hyperbolic differential equations in one space dimension. The optimal control function minimizes an index of performance consisting of the derivatives of the state variable and the control function. Assuming an optimal control exists, a corresponding optimal adjoint variable was introduced that was related to the optimal response function via terminal conditions. A Hamiltonian functional was then introduced involving the optimal adjoint variable and an admissible control function. The maximum principle derived stated that amongst all admissible controls the one that maximized the given Hamiltonian was the optimal control.

The maximum principle developed can be used to construct the solution of the control problem and is aimed at solving active control problems arising in structural mechanics. Active control is employed to damp out the vibrations of structural elements and the performance index used in this study can represent the potential and kinetic energy of the system. The present approach leads to a boundary/terminal/initial value problem, the solution of which can be sought by analytical or computational techniques. It has been shown that the results are applicable to a specific structural dynamics problem, i.e. to the case of the Timoshenko beam.

The effectiveness of the control is illustrated by solving a numerical example involving a beam undergoing flexural and torsional vibrations. The behaviors of the deformations, their velocities as well as the energy are compared for controlled and uncontrolled beams as a function of time. It is observed that the above quantities decrease as a damped sinusoidal function as time increases. In particular, the velocities and the energy of the controlled become substantially smaller compared to the corresponding quantities of the uncontrolled beam, indicating the effectiveness of the proposed control mechanism in damping out the vibrations of structural systems.

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