A Robust Nonlinear Adaptive Backstepping Controller for a CSTR

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Nonlinear backstepping is a recursive design methodology that makes use of the Lyapunov stability theory. Although backstepping can be applied to a larger class of systems than other differential−geometric methods such as feedback linearization, its applicability is limited to "parametric pure-feedback systems". In this work, we apply the idea of backstepping to a benchmark chemical reactor by using a simple transformation of the original nonlinear model of the chemical reactor. This chemical reactor does not fall under the category of systems for which backstepping can be applied. However, the fundamental idea involved in backstepping can still be applied to this process after a certain transformation of the original variables. A robust adaptive nonlinear controller is also designed by introducing uncertainty into all of the estimated parameters. This type of uncertainty leads to nonaffine uncertain parameters that are difficult to handle with the traditional backstepping algorithm. Using Lyapunov theory, we derive a controller that can ensure robust stability.

1. Introduction

"Nonlinearities are the rule rather than exception in chemical processes" was the opening statement in the review article by Kravaris and Kantor.[1] The fact is that virtually all processes are nonlinear. Many common process control problems exhibit nonlinear behavior, in that the relationship between the controlled and manipulated variables depends on the operating conditions. Nevertheless, chemical processes are traditionally controlled by using linear system design techniques, see, e.g., refs 1−4. The dynamic behavior of a process can be approximated by a linear model around a certain operating point. If the process is only mildly nonlinear or remains in the vicinity of a nominal steady state, then the effects of the nonlinearities might not be severe. In that case, conventional linear control design techniques are generally deemed to be adequate. A major reason to rely on linear control is the relative simplicity and computational feasibility of linear control schemes. In recent times, there has been a growing interest in nonlinear process control, especially with the advent of new techniques to address nonlinear control design problems. The most commonly used approaches are feedback linearization, input−output decoupling, and disturbance decoupling. These techniques obviously form a small portion of methods that can be applied in the area of nonlinear control; see, for instance, refs 4 and 5.

During the past five decades, a number of attempts have been made to build controllers that can stabilize and track set points in the presence of uncertainties in the process parameters (not necessarily time-varying). The most elementary feedback loops that were designed in early 1950s can tolerate significant uncertainties. With the advances in modern control theory,[6] it was realized that there is a need for more advanced controllers for uncertain plants to achieve optimal performance. An offshoot of this realization is the development of adaptive linear control theory and the robust control theory. Adaptive linear control is a technique in which the plant model and, as a result, the optimal linear controller are updated continuously using online identification and control algorithms. Many general features of the stability and asymptotic convergence of adaptive linear controllers have been established, and a number of books have been written on adaptive linear control.[7−10] Robust control theory for linear systems was developed assuming that a frequency-domain description of the uncertain bounds is known,[11,12] but the identification of systems with frequency-domain uncertainty descriptions has proved to be difficult. Moreover, most of these techniques can handle only linear systems.

Recent advances in the use of differential−geometric tools for nonlinear feedback control[4,5] have spurred a renewed interest in the adaptive control of nonlinear systems with uncertain parameters. Differential geometric methods have yielded a number of aesthetically pleasing results such as feedback linearization. However, it soon became clear that these methods cannot handle uncertain/unknown parameters. This deficit led researchers to take recourse to some of the old techniques of proving stability using Lyapunov functions. Lyapunov-function-based controller design has, historically, played a very important role in nonlinear control design. Until recently, its applicability was restricted to linear plants with relative degrees of 1 or 2. A new recursive design procedure called *backstepping* has removed this limitation and paved the way to the design of controllers for higher-order nonlinear systems.[13,14]

Backstepping uses a systematic approach for the construction of feedback control laws as well as Lyapunov functions. This removes the difficulties encountered in obtaining a control Lyapunov function for higher-order systems. Backstepping builds into the system a number of strong properties of global or local stability and tracking in a number of recursive steps that never exceeds the system order. Whereas feedback linearization methods require precise models and often cancel useful nonlinearities, backstepping designs offer a choice of design tools for handling uncertain nonlinearities and can avoid wasteful cancellations of nonlinearities.[14]

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Despite its many advantages, backstepping has some drawbacks in that this method cannot be applied to any general nonlinear system (the largest class of systems for which this method can be applied is not yet known). It is well-known that backstepping can be applied to "parametric pure-feedback systems" and "parametric strict-feedback systems" with either the *matching condition* or the *extended matching condition*. Adaptive backstepping can result in overparametrization, but in some cases, this problem can be avoided by using *tuning functions*.

The dynamics and control of a number of chemical reactors have been studied in the literature.[15] In this study, the backstepping method is applied to a model of a continuous stirred tank reactor (CSTR) described in refs 16 and 17. This reactor was designated as a benchmark problem for nonlinear control system design in 17. This CSTR problem exhibits a number of interesting features such as poor performance of the linearized model and unstable zero dynamics. It is also a process with great practical importance. A dynamic state feedback controller that achieves input−output feedback linearization was designed for the same CSTR in ref 18.

This paper is organized as follows: Section 2 provides a detailed description of the process. In section 3, the control problem is posed, and the nonapplicability of static feedback linearization to this particular system is shown. The backstepping controller design methodology is then discussed in section 4. A modified adaptive backstepping algorithm is presented in section 4.4. Finally, a robust adaptive controller is derived in section 5. Simulation results and some explanations for the results are presented in section 6. The last section, section 7, provides conclusions.

2. Process Model Description

In many chemical processes, the main reaction that yields the desired product is accompanied by consecutive and parallel reactions that produce undesired byproducts. An example of a general reaction scheme of this type was used by Kantor[19] to discuss the applicability of extended linearization techniques to the control of reactions in continuous stirred tank reactors. The same reaction mechanism was used by Engell and Klatt[16] to design a gain scheduling controller. This reaction mechanism was attributed to van de Vusse[20] and can be written as

$$\mathrm{A} \xrightarrow{k_1} \mathrm{B} \xrightarrow{k_2} \mathrm{C}$$

$$2\mathrm{A} \xrightarrow{k_3} \mathrm{D}$$

where A is the reactant, B is the desired product, and C and D are the unwanted byproducts. In general, one attempts to make the reaction rates k_2 and k_3 small in comparison to k_1 by an appropriate choice of catalyst and reaction conditions. However, if this is not possible or if it is only possible to a certain extent, the concentration of B in the product stream that leaves the reactor can be controlled by the inflow to the reactor and/or the reaction temperature.

The process considered here involves the production of cyclopentenol (B) from cyclopentadiene (A) by acid-catalyzed electrophilic addition of water in dilute solution. Because of the strong reactivity of the reactant A and the product B, dicyclopentadiene (D) is produced by a Diels−Alder reaction as a side product, and cyclopentanediol (C) is generated as a consecutive product through the addition of another water molecule.[16] The complete reaction scheme is

Table 1. Process Parameters

symbol	value	symbol	value
k_{10}	3.575×10^8 s^{-1}	d_3	−41.85 kJ/(mol of A)
k_{20}	3.575×10^8 s^{-1}	ρ	0.9342 kg/L
k_{30}	2.512×10^6 mol of A s^{-1}	C_ρ	3.01 kJ/(kg K)
E_1	−9758.3 K	C_1	85.6347×10^{-4} s^{-1}
E_2	−9758.3 K	C_2	0.2408 kJ/(s K)
E_3	−8560 K	m	10.0 kJ/K
d_1	4.20 kJ/(mol of A)	V_R	0.01 m^3
d_2	−11.0 kJ/(mol of B)		

$$\mathrm{C_5H_6} \xrightarrow{+\mathrm{H_2O/H^+}} \mathrm{C_5H_7OH} \xrightarrow{\mathrm{H_2O/H^+}} \mathrm{C_5H_8(OH)_2}$$

$$2\mathrm{C_5H_6} \rightarrow \mathrm{C_{10}H_{12}}$$

The dynamics of the reactor can be described by the following set of differential equations (see ref 17 for further details about modeling and physicochemical properties)

$$\dot{c}_\mathrm{A} = \frac{\dot{V}}{V_\mathrm{R}}(c_{\mathrm{A}0} - c_\mathrm{A}) - k_1(\nu)c_\mathrm{A} - k_3(\nu){c_\mathrm{A}}^2 \tag{1a}$$

$$\dot{c}_\mathrm{B} = -\frac{\dot{V}}{V_\mathrm{R}}c_\mathrm{B} + k(\nu)c_\mathrm{A} - k_2(\nu)c_\mathrm{B} \tag{1b}$$

$$\dot{\nu} = \frac{\dot{V}}{V_\mathrm{R}}(\nu_0 - \nu) + C_1(\nu_\mathrm{K} - \nu) - \frac{1}{\rho C_\rho}[k_1(\nu)c_\mathrm{A}d_1 + k_2(\nu)c_\mathrm{B}d_2 + k_3(\nu){c_\mathrm{A}}^2 d_3] \tag{1c}$$

$$\dot{\nu}_\mathrm{K} = \frac{1}{m}\dot{Q}_\mathrm{K} + \frac{C_2}{m}(\nu - \nu_\mathrm{K}) \tag{1d}$$

The concentrations of A and B are c_A and c_B, respectively ($c_\mathrm{A} \geq 0$, $c_\mathrm{B} \geq 0$). The temperature in the reactor is denoted by ν, and the temperature in the cooling jacket is given by ν_K, with both temperatures being expressed on the absolute temperature scale (i.e., in Kelvin). The reaction velocities are assumed to depend on the temperature via the Arrhenius law

$$k_i(\nu) = k_{i0}\exp\left(\frac{E_i}{\infty}\right), \quad i = 1, 2, 3 \tag{2}$$

The inflow of the reactor is composed of only substance A, with the inflow concentration and temperature given by $c_{\mathrm{A}0}$ and ν_0, respectively. Values for the physical and chemical parameters are given in Table 1. Before we proceed with the formulation of our control problem, we take a look at the local or approximate steady-state map of this system. Steady-state maps of this system are shown in Figure 1. The steady-state map between the concentrations of c_A and c_B is an inverted parabola, and that between the temperature in the jacket and the temperature in the reactor is a straight line.

3. Feedback Linearization of a CSTR

We proceed with the formulation of the control problem. The aim is to regulate the concentration of product B and simultaneously regulate the ratio $c_\mathrm{B}/c_\mathrm{A}$. This ratio can be interpretted as a measure of the

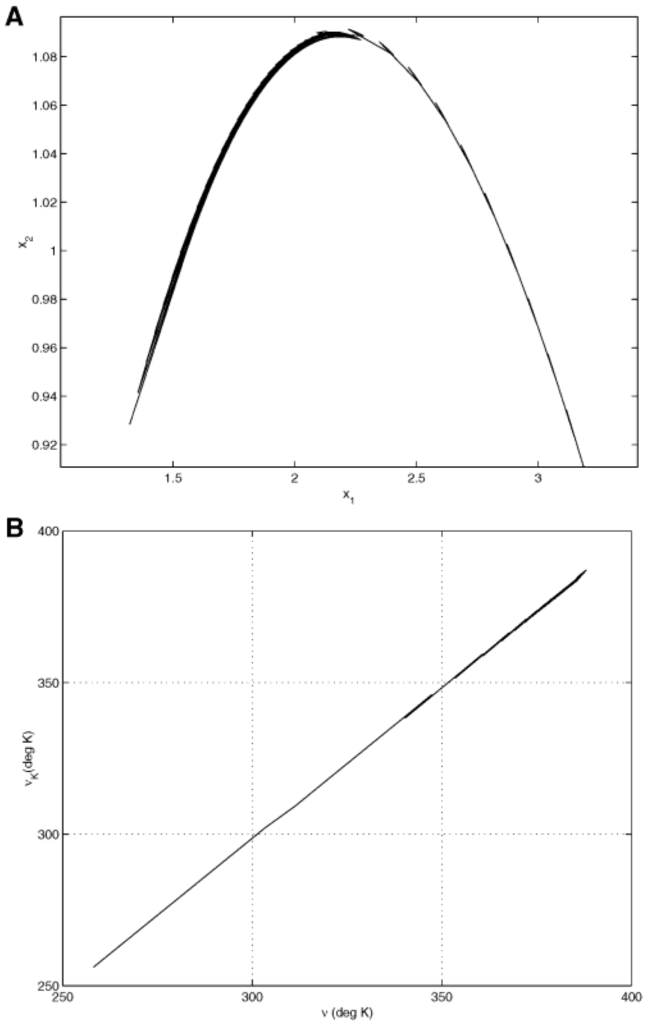

Figure 1. Approximate steady-state maps of the system: (a) c_A vs c_B, (b) ν vs ν_k.

effectiveness of the reaction, and it is certainly useful in regulating the performance of the reactor. We make the following choice of state variables

$$x = [c_A \quad c_B \quad \nu \quad \nu_K]^T \tag{3}$$

Under this choice of state variables, the dynamics of the system takes the form

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 \tag{4}$$

where $u_1 = \dot{V}/V_R$ and $u_2 = \dot{Q}_K$ are the manipulated variables, the vector field $f(x)$ is given by

$$f(x) = \begin{bmatrix} -k_1(x_3)x_1 - k_3(x_3)x_1^2 \\ k_1(x_3)x_1 - k_2(x_3)x_2 \\ C_1(x_4 - x_3) - \dfrac{1}{\rho C_\rho} f_{x_3} \\ \dfrac{C_2}{m}(x_3 - x_4) \end{bmatrix} \tag{5}$$

with

$$f_{x_3} = k_1x_1d_1 + k_2x_2d_2 + k_3x_1^2d_3 \tag{6}$$

the vector fields $g_1(x)$ and $g_2(x)$ are given by

$$g_1(x) = \begin{bmatrix} (c_{A0} - x_1) \\ -x_2 \\ (\nu_0 - x_3) \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{m} \end{bmatrix} \tag{7}$$

and the output functions are

$$y_1(x) = h_1(x) = x_2, \quad y_2(x) = h_2(x) = \frac{x_2}{x_1} \tag{8}$$

We also assume that the state variables satisfy the following conditions

$$x_1 > 0, \quad x_2 > 0, \quad x_3 > 0 \tag{9}$$

As a first step, the relative degree of this system needs to be calculated to check the possibility of simplifying the model equations through some coordinate transformation.

The characteristic numbers or the relative degrees of the system in eq 4 are $r_1 = 1$ and $r_2 = 1$, since $L_{g1}h(x) \neq 0$ and $L_{g2}h(x) = 0$ for both of the outputs. Therefore, after differentiating the outputs once, the following equation is obtained

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = F(x) + G(x)u \tag{10}$$

where $u = [u_1\ u_2]^T$ and the matrix $G(x)$ is the decoupling matrix, which is given by

$$G(x) = \begin{bmatrix} -x_2 & 0 \\ -c_{A0}\dfrac{x_2}{x_1^2} & 0 \end{bmatrix} \tag{11}$$

Because the decoupling matrix is singular, we can conclude that the original system in eq 4 cannot be input–output decoupled by means of static feedback.[4] In ref 18, a dynamic state feedback controller was used to input–output linearize the system. It was also shown that the compensated system was equivalent to a linear system or, in other words, had a trivial zero dynamics.[4,5] Their method did not take into account the boundedness of inputs, outputs, and state signals or parameter and model uncertainties. In fact, their simulation results clearly show that the inputs, inlet flow rate, and heat removal rate are too large in the vicinity of a step change in the set points. Moreover, feedback linearization cannot guarantee either asymptotic tracking or global stability in the presence of uncertain parameters. As described earlier, the natural choice of controller that can address all of the above issues is a controller designed using the idea of backstepping. Before we proceed with the formulation of the backstepping problem for this system, we note that this system can be input–output linearized by a different choice of outputs.

For example, if

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \tag{12}$$

then the decoupling matrix is given by

$$D^1(x) = \begin{bmatrix} -x_2 & 0 \\ 0 & \frac{1}{m} \end{bmatrix} \quad (13)$$

which is clearly nonsingular, and therefore, the system can be input−output linearized by means of a static state feedback.[4]

4. Backstepping Controller Design

Backstepping is a popular new controller design technique for uncertain plants with a certain model structure. Although the general proofs are restricted to plants with some special structures, the same idea can be used for other structures also. The largest general structure of the types of systems on which backstepping can be applied is not yet known. Although the idea of backstepping is new, it is based on well-known concepts of Lyapunov functions and control Lyapunov functions. Backstepping simplifies the problem of finding the control Lyapunov function (clf) for a multistate system into that of finding the clf for a subsystem with a smaller number of states (usually one) by creating a number of virtual control signals.

It has been shown that state feedback backstepping can be applied to systems of the so-called class of parametric pure-feedback systems and to its subclass of parametric strict-feedback systems. The class of pure-feedback systems with unknown parameters is well represented by the following third-order system[14]

$$\begin{aligned} \dot{x}_1 &= x_2 + \varphi_1^{\mathrm{T}}(x_1,x_2)\theta \\ \dot{x}_2 &= x_3 + \varphi_2^{\mathrm{T}}(x_1,x_2,x_3)\theta \\ \dot{x}_3 &= u + \varphi_3^{\mathrm{T}}(x_1,x_2,x_3)\theta \end{aligned} \quad (14)$$

where the $p \times 1$ vector θ is constant and unknown. The pure-feedback systems are characterized by the affine dependence of θ and the structures of the known nonlinearities φ_1, φ_2, and φ_3. The function φ_1 must not depend on x_3, but it should depend on x_2; likewise, φ_2 must depend on x_3. The particular CSTR studied here does not fall into this class of systems, but the same idea can be used.

4.1. Problem Formulation. Using the same notation as in section 3 for the states, the nonlinear system can be represented as follows

$$\dot{x}_1 = u_1(c_{A0} - x_1) - k_1(x_3)x_1 - k_3(x_3)x_1^2 \quad (15a)$$

$$\dot{x}_2 = -u_1x_2 + k_1(x_3)x_1 - k_2(x_3)x_2 \quad (15b)$$

$$\dot{x}_3 = u_1(\nu_0 - x_3) + C_1(x_4 - x_3) - \frac{1}{\rho C_\rho}[k_1(x_3)x_1d_1 + k_2(x_3)x_2d_2 + k_3(x_3)x_1^2d_3] \quad (15c)$$

$$\dot{x}_4 = \frac{1}{m}u_2 + \frac{C_2}{m}(x_3 - x_4) \quad (15d)$$

where $k_i(x_3) = k_{i0}e^{E_i/x_3}$ and k_{i0} and E_i are constants. For the system of eqs 15a−15d, there is only one disturbance (ν_0) and two inputs. The control objective is to allow the two outputs to track two separate reference trajectories with bounded inputs. It is important to note that we cannot take the system to any arbitrary state (x_1, x_2). This is clear from Figure 1, as no choice of inputs can take the system to a point inside or outside the parabola, or in other words, it is not possible to choose a steady-state operating point that does not lie on the parabola.

4.2. Backstepping Controller. The system of eqs 15a−15d does not have the parametric pure-feedback structure, but the idea of backstepping can still be applied to this system by a judicious choice of virtual inputs. Equation 15a contains two states along with an actual input. Let us assume that all of the states are available from measurements (state feedback). Let the reference trajectory for the first state, x_1, be x_{1r}. The error system can be written as

$$\dot{z}_1 = \dot{x}_1 - \dot{x}_{1r} = u_1(c_{A0} - x_1) - k_1(x_3)x_1 - k_3(x_3)x_1^2 - \dot{x}_{1r} \quad (16)$$

where $z_1 = x_1 - x_{1r}$. It is assumed that the first derivative of the reference signal exists and is well-defined. Then, the following choice of input u_1 and control Lyapunov function V_0 would render the subsystem in eq 16 globally asymptotically stable

$$u_1 = \frac{-c_1z_1 + k_1x_1 + k_3x_1^2 + \dot{x}_{1r}}{c_{A0} - x_1} \quad (17)$$

$$V_0 = \frac{1}{2}z_1^2 \quad (18)$$

where c_1 is a positive constant.

Note that the denominator of the first input, u_1, can never be zero and, therefore, the input remains finite, although it might not be within practically feasible limits. Now, substituting the input into the error expression in eq 16, we obtain

$$\dot{z}_1 = -c_1z_1$$

Therefore

$$\begin{aligned} \dot{V}_0 &= z_1\dot{z}_1 \\ &= -c_1z_1^2 \end{aligned} \quad (19)$$

$$\rightarrow$$

$$\dot{V}_0 < 0 \quad \forall c_1 > 0, z_1 \neq 0$$

As a result, the subsystem in eq 15a is globally asymptotically stable. Now, we take a step back and consider the error system for the second state

$$\dot{z}_2 = \dot{x}_2 - \dot{x}_{2r} = -u_1x_2 + k_1(x_3)x_1 - k_2(x_3)x_2 - \dot{x}_{2r} \quad (20)$$

Because we have already chosen the first input u_1, we do not have any leverage over that input. We cannot manipulate x_1, and we cannot take this state to an arbitrary value because it has to follow a trajectory. That leaves us with only one possibility, i.e., take the third state, x_3, to a desired value such that x_2 asymptotically tracks the reference signal (x_{2r}). However, the third state, reactor temperature, cannot be manipulated directly. It can only be changed indirectly by manipulating the second input, u_2. The virtual control x_{3r} that lets the second state track its reference trajectory is called a stabilizing function.

Our aim is to augment the original Lyapunov function V_0 with the second error variable and find the stabilizing trajectory for x_3. The following trajectory for x_3 makes the subsystem in eqs 16 and 20 globally asymptotically stable

$$x_{3r} = \frac{E_1}{\ln\left(\frac{-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}}{k_{10}x_1 - k_{20}x_2}\right)} \quad (21)$$

where c_2 is a positive constant. This simplification is possible because $E_1 = E_2$. Now, the augmented Lyapunov function is

$$V_1 = V_0 + \frac{1}{2}{z_2}^2 \quad (22)$$

→

$$\begin{aligned}\dot{V}_1 &= z_1\dot{z}_1 + z_2\dot{z}_2\\ &= -c_1{z_1}^2 - c_2{z_2}^2\end{aligned} \quad (23)$$

→

$$\dot{V}_1 < 0 \quad \forall c_1, c_2 > 0, (z_1, z_2) \neq 0$$

The subsystem in eqs 16 and 20 is globally asymptotically stable with the virtual input in eq 21 and the clf in eq 22, but if $x_3 \neq x_{3r}$, then nothing can be said about the asymptotic stability of this subsystem. It will be shown in the following paragraphs that, by making x_3 and x_4 track their respective trajectories arbitrarily closely, global asymptotic stability can be achieved.

The virtual input x_{3r} can be manipulated by adjusting another virtual input, x_4, through the second input. The third error system is then

$$\begin{aligned}\dot{z}_3 &= \dot{x}_3 - \dot{x}_{3r}\\ &= u_1(\nu_0 - x_3) + C_1(x_4 - x_3) - \frac{1}{\rho C_\rho} f_{x_3} - \dot{x}_{3r}\end{aligned} \quad (24)$$

where f_{x_3} is defined by eq 6.

Now choose

$$x_{4r} = \frac{1}{C_1}\left(-c_3 z_3 - u_1(\nu_0 - x_3) + C_1 x_3 + \frac{1}{\rho C_\rho} f_{x_3} + \dot{x}_{3r}\right) \quad (25)$$

Therefore, if x_{4r} were the true input, the derivative of the following Lyapunov function would be negative definite

$$V_3 = \frac{1}{2}{z_3}^2 \quad (26)$$

→

$$\dot{V}_3 = -c_3{z_3}^2 \quad (27)$$

However, because x_{4r} is not the true input

$$\dot{V}_3 = -c_3{z_3}^2 + C_1 z_3 z_4 \quad (28)$$

The fourth error system is then

$$\begin{aligned}\dot{z}_4 &= \dot{x}_4 - \dot{x}_{4r}\\ &= \frac{1}{m}u_2 + \frac{C_2}{m}(x_3 - x_4) - \dot{x}_{4r}\end{aligned} \quad (29)$$

Choosing the input

$$u_2 = -c_4 m z_4 - C_2(x_3 - x_4) - C_1 m z_3 + m\dot{x}_{4r} \quad (30)$$

we obtain

$$\dot{z}_4 = -c_4 z_4 - C_1 z_3 \quad (31)$$

The subsystem in eqs 24 and 29 will globally asymptotically track any two trajectories x_{3r} and x_{4r} with the Lyapunov function

$$V_4 = \frac{1}{2}{z_3}^2 + \frac{1}{2}{z_4}^2 \quad (32)$$

and its derivative

$$\dot{V}_4 = -c_3{z_3}^2 - c_4{z_4}^2 < 0 \quad \forall c_3, c_4 > 0, (z_3, z_4) \neq (0, 0) \quad (33)$$

Therefore, in due course of time, regardless of the reference trajectory x_{3r}, the errors z_3 and z_4 can be made arbitrarily small by manipulating the second input, u_2. As mentioned earlier, because x_{3r} is only a virtual input, x_3 will never be exactly x_{3r}, and therefore, eq 23 has to be modified. Substituting $z_3 + x_{3r}$ for x_3 into eq 20, we obtain

$$\dot{z}_2 = -u_1 x_2 + k_{10}e^{E_1/z_3+x_{3r}} - k_{20}e^{E_2/z_3+x_{3r}} - \dot{x}_{2r} \quad (34)$$

Note that $\dot{z}_2 \neq -c_2 z_2$ as long as $z_3 \neq 0$. From eq 33, however, z_3 can be made arbitrarily small. Also, note that $\dot{z}_2$ and z_2 are bounded for any arbitrary values of z_3 and x_{3r}. This follows from the facts that u_1 is positive and $k_1(x_3)$ and $k_2(x_3)$ are bounded. Consequently, $\dot{z}_2$ can be made arbitrarily close to $-c_2 z_2$. As a result, eq 23 will become negative definite after a finite interval of time has elapsed, and it will remain negative definite thereafter. Thus, we are able to prove that, with the virtual inputs (stabilizing functions) defined above, we can achieve a globally asymptotically tracking controller. In summary, the controller is

$$\begin{aligned}u_1 &= \frac{-c_1 z_1 + k_1 x_1 + k_3 {x_1}^2 + \dot{x}_{1r}}{c_{A0} - x_1}\\ x_{3r} &= \frac{E_1}{\ln\left(\frac{-c_2 z_2 + u_1 x_2 + \dot{x}_{2r}}{k_{10}x_1 - k_{20}x_2}\right)}\\ x_{4r} &= \frac{1}{C_1}\left[-c_3 z_3 - u_1(\nu_0 - x_3) + C_1 x_3 + \frac{1}{\rho C_\rho} f_{x_3} + \dot{x}_{3r}\right]\\ u_2 &= -c_4 m z_4 - C_2(x_4 - x_3) - C_1 m z_3 - m\dot{x}_{4r}\end{aligned} \quad (35)$$

4.3. Backstepping without Canceling Useful Nonlinearities. One major advantage of backstepping is that, by a careful choice of stabilizing functions, unnecessary cancellations of useful nonlinearities can be avoided (unlike in feedback linearization).

Replacing x_1 with $z_1 + x_{1r}$ in eq 16, we obtain

$$\dot{z}_1 = -u_1 z_1 - k_1(x_3)z_1 + u_1(c_{A0} - x_{1r}) - k_1(x_3)x_{1r} - k_3(x_3){x_1}^2 \quad (36)$$

Now we can choose

$$u_1 = \frac{-c_1z_1 + k_1x_{1r} + k_3{x_1}^2 + \dot{x}_{1r}}{c_{A0} - x_{1r}} \quad (37)$$

which results in

$$\dot{z}_1 = -c_1z_1 - u_1z_1 - k_1(x_3)z_1 \quad (38)$$

Note that u_1 and $k_1(x_3)$ are always positive. Similarly x_{3r}, x_{4r}, *and* u_2 can be modified to

$$x_{3r} = \frac{E_1}{\ln\left(\dfrac{-c_2z_2 + u_1x_{2r} + \dot{x}_{2r}}{k_{10}x_1 - k_{20}x_{2r}}\right)} \quad (39)$$

$$x_{4r} = \frac{1}{C_1}\left[-c_3z_3 - u_1(\nu_0 - x_{3r}) + C_1x_{3r} + \frac{1}{\rho C_\rho}f_{x_3} + \dot{x}_{3r}\right] \quad (40)$$

$$u_2 = -c_4mz_4 - C_2(x_3 - x_{4r}) - C_1mz_3 - m\dot{x}_{4r} \quad (41)$$

The derivatives in the virtual and true inputs result in large values for the inputs. This situation can be rectified to an extent by another simplification that simply drops the derivatives in the inputs. Dropping the derivatives would still ensure tracking because of the physical nature of the process. We are only interested in tracking approximate step changes in set points; therefore, $\dot{x}_{1r}$ and $\dot{x}_{2r}$ should go to zero very rapidly. Ignoring these derivatives will not cause any problems as far as tracking x_1 and x_2 are concerned. If we ignore the derivatives $\dot{x}_{3r}$ and $\dot{x}_{4r}$, the Lyapunov function V_4 has to be changed to

$$\dot{\tilde{V}}_4 = -c_3{z_3}^2 - c_4{z_4}^2 - \dot{x}_{3r}z_3 - \dot{x}_{4r}z_4 \quad (42)$$

Because $\dot{x}_{1r}$ and $\dot{x}_{2r}$ go to zero, $\dot{x}_{3r}$ would also approach zero eventually, and as a result, $\dot{x}_{4r}$ too approaches zero. Therefore, $\dot{\tilde{V}}_4$ would still be negative definite after a certain finite time. Also, note that, if $\dot{x}_{4r}$ and z_4 have the same sign, they help in reducing the error z_4. This set is also an invariant set: once you are in the set where $\dot{x}_{4r}$ and z_4 have the same sign, you will never leave it. If they have opposite signs, they can increase the Lyapunov function and thus the errors, but the nature of the process and the output tracking requirements do not allow this possibility, and therefore, $\dot{x}_{4r}$ has to change its sign after some finite time and enter the above invariant set. These arguments apply to the other term, $\dot{x}_{3r}z_3$, as well. In the rest of the analysis, these derivatives are ignored as they are found to be harmless. Of course, this can have a detrimental effect on the transient performance but that is the price of trying to obtain bounded inputs.

4.4. Backstepping on the Transformed Model. The system of eqs 15a−15d does not have the parametric pure-feedback structure, but the idea of backstepping can still be applied to this system by a judicious choice of virtual inputs after a certain transformation of the original variables. Equation 15a contains two states along with an actual input. Let us assume that all of the states are available from measurements (state feedback). The elegance of the backstepping approach is destroyed by the presence of the term e^{E_i/x_3} in the second ODE. An interesting transformation can be used to avoid this problem. Let us define a new state

$$\tilde{x}_3 = e^{E_1/x_3} \quad (43)$$

Then

$$\dot{\tilde{x}}_3 = -\tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}\dot{x}_3 \quad (44)$$

With this transformation, the original system can be rewritten as

$$\dot{x}_1 = u_1(c_{A0} - x_1) - k_{10}\tilde{x}_3x_1 - k_{30}{\tilde{x}_3}^{E_3/E_1}{x_1}^2 \quad (45)$$

$$\dot{x}_2 = -u_1x_2 + k_{10}\tilde{x}_3x_1 - k_{20}\tilde{x}_3x_2 \quad (46)$$

$$\dot{\tilde{x}}_3 = -\tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}\left[u_1\left(\nu_0 - \frac{E_1}{\ln \tilde{x}_3}\right) + C_1\left(x_4 - \frac{E_1}{\ln \tilde{x}_3}\right) - \frac{1}{\rho C_\rho}f_{x_3}\right] \quad (47)$$

$$\dot{x}_4 = \frac{1}{m}u_2 + \frac{C_2}{m}\left(\frac{E_1}{\ln \tilde{x}_3} - x_4\right) \quad (48)$$

Step 1. Choose

$$u_1 = \frac{-c_1z_1 + k_{10}\tilde{x}_3x_1 + k_{30}\tilde{x}^{E_3/E_1}{x_1}^2 + \dot{x}_{1r}}{c_{A0} - x_1} \quad (49)$$

and the Lyapunov function

$$V_0 = \frac{1}{2}{z_1}^2 \quad (50)$$

→

$$\dot{V}_0 = z_1\dot{z}_1 = -c_1{z_1}^2 < 0 \quad \forall c_1 > 0,\ z_1 \neq 0 \quad (51)$$

Step 2. The second error variable is

$$\dot{z}_2 = -u_1x_2 + k_{10}\tilde{x}_3x_1 - k_{20}\tilde{x}_3x_2 - \tilde{x}_{2r} \quad (52)$$

Augment the Lyapunov function V_0 to obtain

$$V_1 = V_0 + \frac{1}{2}{z_2}^2 \quad (53)$$

→

$$\dot{V}_1 = z_1\dot{z}_1 + z_2\dot{z}_2 = -c_1{z_1}^2 + z_2(-u_1x_2 + k_{10}\tilde{x}_3x_1 - k_{20}\tilde{x}_3x_2 - \tilde{x}_{2r}) \quad (54)$$

Now, choosing

$$\tilde{x}_{3r} = \frac{-c_2z_2 + u_1x_2 + \dot{x}_{2r}}{(k_{10}x_1 - k_{20}x_2)} \quad (55)$$

and replacing $\tilde{x}_3$ by $z_3 + \tilde{x}_{3r}$ in eq 54, we obtain

$$\dot{V}_1 = -c_1{z_1}^2 + z_2(-u_1x_2 + k_{10}\tilde{x}_{3r}x_1 - k_{20}\tilde{x}_{3r}x_2 - \tilde{x}_{2r}) + z_2z_3(k_{10}x_1 - k_{20}x_2) = -c_1{z_1}^2 - c_2{z_2}^2 + z_2z_3(k_{10}x_1 - k_{20}x_2) \quad (56)$$

where c_2 is a positive constant. $\dot{V}_1$ is clearly not negative definite. We attempt to cancel the indefinite term

$z_2z_3(k_{10}x_1 - k_{20}x_2)$ in the next augmentation step. Observe that the denominator of $\tilde{x}_{3r}$ is never zero (see Figure 1a).

Step 3. The third error variable is

$$\dot{z}_3 = \tilde{x}_3 - \tilde{x}_{3r} \tag{57}$$

$$= -\tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}\left[u_1\left(\nu_0 - \frac{E_1}{\ln \tilde{x}_3}\right) + C_1\left(x_4 - \frac{E_1}{\ln \tilde{x}_3}\right) - \frac{1}{\rho C_\rho} f_{\tilde{x}_3}\right] - \tilde{x}_{3r} \tag{58}$$

Augment V_1 to

$$\dot{V}_2 = V_1 + \frac{1}{2}z_3^{\ 2} \tag{59}$$

→

$$\dot{V}_2 = -c_1z_1^{\ 2} - c_2z_2^{\ 2} + z_2z_3(k_{10}x_1 - k_{20}x_2) + z_3\dot{z}_3$$
$$= -c_1z_1^{\ 2} - c_2z_2^{\ 2} + z_2z_3(k_{10}x_1 - k_{20}x_2) + z_3\left[-\tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}\left(u_1\left(\nu_0 - \frac{E_1}{\ln \tilde{x}_3}\right) + C_1\left(x_4 - \frac{E_1}{\ln \tilde{x}_3}\right) - \frac{1}{\rho C_\rho} f_{\tilde{x}_3}\right) - \tilde{x}_{3r}\right] \tag{60}$$

Choosing

$$x_{4r} = \frac{1}{C_1}\left[-u_1\left(\nu_0 - \frac{E_1}{\ln \tilde{x}_3}\right) + C_1\frac{E_1}{\ln \tilde{x}_3} + \frac{1}{\rho C_\rho} f_{\tilde{x}_3}\right] + \frac{1}{-\tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}C_1}[-c_3z_3 - z_2(k_{10}x_1 - k_{20}x_2) + \tilde{x}_{3r}] \tag{61}$$

Substituting $z_4 + x_{4r}$ for x_4, we obtain

$$\dot{V}_2 = -c_1z_1^{\ 2} - c_2z_2^{\ 2} - c_3z_3^{\ 2} - \tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}C_1z_3z_4 \tag{62}$$

Step 4. The fourth error variable is

$$\dot{z}_4 = \dot{x}_4 - \dot{x}_{4r}$$
$$= \frac{1}{m}u_2 + \frac{C_2}{m}\left(\frac{E_1}{\ln \tilde{x}_3} - x_4\right) - \dot{x}_{4r} \tag{63}$$

Augmenting the Lyapunov function V_2, we obtain

$$V_3 = V_2 + \frac{1}{2}z_4^{\ 2} \tag{64}$$

→

$$\dot{V}_3 = -c_1z_1^{\ 2} - c_2z_2^{\ 2} - c_3z_3^{\ 2} - \tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}C_1z_3z_4 + z_4\dot{z}_4$$
$$= -c_1z_1^{\ 2} - c_2z_2^{\ 2} - c_3z_3^{\ 2} - \tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}C_1z_3z_4 + z_4\left[\frac{1}{m}u_2 + \frac{C_2}{m}\left(\frac{E_1}{\ln \tilde{x}_3} - x_4\right) - \dot{x}_{4r}\right] \tag{65}$$

Choosing

$$u_2 = m\left[-c_4z_4 + \tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}C_1z_3 - \frac{C_2}{m}\left(\frac{E_1}{\ln \tilde{x}_3} - x_4\right) + \dot{x}_{4r}\right] \tag{66}$$

we obtain

$$\dot{V}_3 = -c_1z_1^{\ 2} - c_2z_2^{\ 2} - c_3z_3^{\ 2} - c_4z_4^{\ 2} < 0 \tag{67}$$

$$\forall c_i > 0 \quad (z_1, z_2, z_3, z_4) \neq (0, 0, 0, 0) \tag{68}$$

In summary, the controller is

$$u_1 = \frac{-c_1z_1 + k_{10}\tilde{x}_3x_1 + k_{30}\tilde{x}^{E_3/E_1}x_1^{\ 2} + \dot{x}_{1r}}{c_{A0} - x_1}$$

$$\tilde{x}_{3r} = \frac{-c_2z_2 + u_1x_1 + \dot{x}_{2r}}{(k_{10}x_1 - k_{20}x_2)}$$

$$x_{4r} = \frac{1}{C_1}\left[-u_1\left(\nu_0 - \frac{E_1}{\ln \tilde{x}_3}\right) + C1\frac{E_1}{\ln \tilde{x}3} + \frac{1}{\rho C_\rho}f_{\tilde{x}_3}\right] + \frac{1}{-\tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}C_1}[-c_3z_3 - z_2(k_{10}x_1 - k_{20}x_2) + \tilde{x}_{3r}]$$

$$u_2 = m\left[-c_4z_4 - C_1z_3 - \frac{C_2}{m}\left(\frac{E_1}{\ln \tilde{x}_3} - x_4\right) + \dot{x}_{4r}\right] \tag{69}$$

and the closed-loop system is

$$\dot{z}_1 = -c_1z_1$$
$$\dot{z}_2 = -c_2z_2 + z_3(k_{10}x_1 - k_{20}x_2)$$
$$\dot{z}_3 = -c_3z_3 - z_2(k_{10}x_1 - k_{20}x_2) - \tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}C_1z_4$$
$$\dot{z}_4 = -c_4z_4 + \tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}C_1z_3 \tag{70}$$

or, in matrix form

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\end{bmatrix} = \begin{bmatrix}-c_1 & 0 & 0 & 0\\ 0 & -c_2 & (k_{10}x_1 - k_{20}x_2) & 0\\ 0 & -(k_{10}x_1 - k_{20}x_2) & -c_3 & -\tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}C_1\\ 0 & 0 & \tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E_1}C_1 & -c_4\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\end{bmatrix}$$

As with any other closed-loop system with a backstepping controller, the "system" matrix has a negative diagonal and is skew symmetric.

4.5. Adaptive Backstepping. The power of backstepping lies not in being able to design a controller for large-dimensional problems but in controlling plants with uncertain parameters. In the model in eqs 1, there are a number of estimated parameters. Invariably, all of the estimated parameters have some uncertainty associated with them. Please note that the derivatives in the following paragraphs are numerically approximated to make the simulations possible. In the next section, a robust adaptive controller that takes into account uncertainties in all of the estimated parameters is derived. The following four uncertain parameters[16] are used to derive the adaptive backstepping controller

(a) $k_{10} = k_{20} = (3.575 \pm 1.11) \times 100^8$

(b) $k_{30} = (2.512 \pm 0.075) \times 10^6$

(c) $C_p = 3.01 \pm 0.04$

(d) $\rho = 0.9342 \pm (4 \times 10^{-4})$

In this section, we consider only the first two uncertain parameters for the sake of simplicity in exposition.

Step 1. Let us represent the estimated values of k_{10} and k_{30} by $\tilde{k}_{10}10^8$ and $\tilde{k}_{30}10^6$, where $\tilde{k}_{10}$ and $\tilde{k}_{30}$ are the uncertain parameters. Henceforth, we assume that $\tilde{k}_{10} = \tilde{k}_{20}$. A number of cancellations that we carried out in the previous section are not possible if the parameters are uncertain. Substituting the input in eq 49 with estimated parameters into $\dot{z}_1$, given by

$$\dot{z}_1 = u_1(c_{A0} - x_1) - k_{10}10^8\tilde{x}_3x_1 - k_{30}10^6\tilde{x}^{E_3/E_1}\,x_1^{\;2} - \dot{x}_{1r} \quad (71)$$

we obtain

$$\dot{z}_1 = -c_1z_1 + (\tilde{k}_{10} - k_{10})10^8\tilde{x}_3x_1 + (\tilde{k}_{30} - k_{30})10^6\tilde{x}^{E_3/E_1}\,x_1^{\;2} \quad (72)$$

Consider the modified Lyapunov function

$$V_0 = \frac{1}{2}z_1^{\;2} + \frac{1}{2\gamma_1^{\;2}}(\tilde{k}_{10} - k_{10})^2 + \frac{1}{2\gamma_2^{\;2}}(\tilde{k}_{30} - k_{30})^2 \quad (73)$$

and its derivative

$$\dot{V}_0 = z_1\dot{z}_1 + \frac{1}{\gamma_1^{\;2}}(\tilde{k}_{10} - k_{10})\dot{\tilde{k}}_{10} + \frac{1}{\gamma_2^{\;2}}(\tilde{k}_{30} - k_{30})\tilde{k}_{30}$$
$$= -c_1z_1^{\;2} + (\tilde{k}_{10} - k_{10})\left(\frac{1}{\gamma_1^{\;2}}\dot{\tilde{k}}_{10} + 10^8\tilde{x}_3x_1z_1\right) + (\tilde{k}_{30} - k_{30})\left(\frac{1}{\gamma_2^{\;2}}\dot{\tilde{k}}_{30} + 10^6\tilde{x}^{E_3/E_1}\,x_1^{\;2}z_1\right) \quad (74)$$

We can now choose

$$\dot{\tilde{k}}_{10} = -\gamma_1^{\;2}10^8\tilde{x}_3x_1z_1$$
$$\dot{\tilde{k}}_{30} = -\gamma_2^{\;2}10^6\tilde{x}^{E_3/E_1}\,x_1^{\;2}z_1 \quad (75)$$

to make the derivative of the Lyapunov function independent of the unknown true parameters k_{10} and k_{30}

$$\dot{V}_0 = -c_1z_1^{\;2}$$

Step 2. Note that we do not have the matching condition, i.e., the uncertain parameter and the virtual input appear as a product. As in the previous case, eq 56 can be written (substituting for $\tilde{x}_{3r}$), after being augmented with $1/\gamma_3^{\;2}(\tilde{k}_{10} - k_{10})^2$, as

$$\dot{V}_1 = -c_1z_1^{\;2} + z_2\left[-u_1x_2 + \frac{k_{10}}{\tilde{k}_{10}}(-c_2z_2 + u_1x_2 + \dot{x}_{2r}) - \dot{x}_{2r}\right] + \frac{1}{\gamma_3^{\;2}}(\tilde{k}_{10} - k_{10})\dot{\tilde{k}}_{10} + z_2z_3(k_{10}x_1 - k_{20}x_2) \quad (76)$$

Choosing

$$\dot{\tilde{k}}_{10} = \dot{\tilde{k}}_{20} = \frac{\gamma_3^{\;2}}{\tilde{k}_{10}}(-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 \quad (77)$$

we obtain

$$\dot{V}_1 = -c_1z_1^{\;2} - c_2z_2^{\;2} + z_2z_3(k_{10}x_1 - k_{20}x_2) \quad (78)$$

Note that these are the parameter update laws for the uncertain parameters in the first virtual input $\tilde{x}_{3r}$ and eqs 75 are the update laws for the unknown parameters in u_1.

Step 3. Augmenting the Lyapunov function V_2 with $1/\gamma_4^{\;2}(\tilde{k}_{10} - k_{10})^2 + 1/\gamma_5^{\;2}(\tilde{k}_{30} - k_{30})^2$, we obtain

$$\dot{V}_2 = -c_1z_1^{\;2} - c_2z_2^{\;2} - c_3z_3^{\;2} + z_2z_3(\tilde{k}_{10} - k_{10})(x_1 - x_2) + \frac{z_3}{\rho C_\rho}[(\tilde{k}_{10} - k_{10})10^8\tilde{x}_3x_1d_1 + (\tilde{k}_{20} - k_{20})10^8\tilde{x}_3x_2d_2 + (\tilde{k}_{30} - k_{30})10^6\tilde{x}_3x_1^{\;2}d_3] + \frac{1}{\gamma_4^{\;2}}(\tilde{k}_{10} - k_{10})\dot{\tilde{k}}_{10} + \frac{1}{\gamma_5^{\;2}}(\tilde{k}_{30} - k_{30})\dot{\tilde{k}}_{30}$$
$$= -c_1z_1^{\;2} - c_2z_2^{\;2} - c_3z_3^{\;2} + (\tilde{k}_{10} - k_{10})[z_2z_3(x_1 - x_2) + \frac{1}{\rho C_\rho}(10^8\tilde{x}_3x_1d_1 + 10^8\tilde{x}_3x_2d_2)z_3 + \frac{1}{\gamma_4^{\;2}}\dot{\tilde{k}}_{10}] + (\tilde{k}_{30} - k_{30})\left[\frac{1}{\gamma_5^{\;2}}\dot{\tilde{k}}_{30} + \frac{1}{\rho C_\rho}(10^6\tilde{x}_3x_1^{\;2}d_3z_3)\right] \quad (79)$$

The following choice of update laws would make the Lyapunov function independent of the unknown parameters

$$\dot{\tilde{k}}_{10} = -\gamma_4^{\;2}\left[z_2z_3(x_1 - x_2) + \frac{1}{\rho C_\rho}(10^8\tilde{x}_3x_1d_1 + 10^8\tilde{x}_3x_2d_2)z_3\right]$$
$$\dot{\tilde{k}}_{30} = -\gamma_5^{\;2}\frac{1}{\rho C_\rho}[10^6\tilde{x}_3x_1^{\;2}d_3z_3] \quad (80)$$

These are the update laws for the virtual input x_{4r}. Including all of the above update laws results in

$$\dot{V}_3 = -c_1z_1^{\;2} - c_2z_2^{\;2} - c_3z_3^{\;2} - c_4z_3^{\;2} \quad (81)$$

Using Lasalle's theorem, it is easy to see that all of the trajectories would go to $(z_1, z_2, z_3, z_4) = (0, 0, 0, 0)$ asymptotically. This method of designing update laws can result in overparametrization, as in this case. There are two uncertain parameters and five update laws. This problem can be avoided by designing tuning functions. At this point, it is important to observe that this method does not guarantee the convergence of uncertain parameters to their true values. It can only guarantee asymptotic tracking of the outputs.

4.6. Tuning Function Design. In the previous section, we tried to eliminate the unknown parameters at every step, but instead, for closed-loop asymptotic tracking, we can wait until the last step and then cancel all of the unknowns with just one update law for each uncertain parameter. The new update law will then be the sum of all of the previous update laws for each uncertain parameter with the same adaptive gain. All of the update laws that comprise the final update law are called tuning functions. The new update laws can be shown to be

$$\dot{\hat{k}}_{10} = -\gamma_1^{\,2}\Big[10^8\tilde{x}_3x_1z_1 + \frac{z^2}{\hat{k}_{10}}(-c_2z_2 + u_1x_2 + \dot{x}_{2r}) + z_2z_3(x_1 - x_2) + \frac{1}{\rho C_\rho}(10^8\tilde{x}_3x_1d_1 + 10^8\tilde{x}_3x_2d_2)z_3\Big] \quad (82)$$

$$\dot{\hat{k}}_{30} = -\gamma_2^{\,2}\left(10^6\tilde{x}^{E_3/E_1}\,x_1^{\,2}z_1 + \frac{1}{\rho C_\rho}10^6\tilde{x}_3x_1^{\,2}d_3z_3\right) \quad (83)$$

5. Robust Adaptive Backstepping

In this section, we assume that all of the estimated constants are uncertain and then derive a controller that is robustly stable and can track the set point despite these uncertainties. Note that k_{i0} is a positive constant and E_i is a negative constant. We also assume that $E_1 = E_2$ and $k_{10} = k_{20}$. Furthermore, suppose that the upper and the lower bounds, E_{01} and E_{02}, respectively, of E_1 are known, i.e., E_{01} and E_{02} are such that $E_{02} \le E_2 = E_1 \le E_{01} < 0$. Because both E_i's are negative constants the $k_i(x_3)$ values are bounded for all $x_3 \ge 0$. Now, considering the following new variable with an arbitrarily chosen $E > E_{01}$ (E can be considered as a tuning parameter) and a scaling factor k_n

$$\tilde{x}_3 = k_{\rm n}{\rm e}^{E/x_3}$$

the controlled plant can be written as

$$\dot{x}_1 = u_1(c_{\rm A0} - x_1) - k_1(\tilde{x}_3)x_1 - k_3(\tilde{x}_3)x_1^{\,2} \quad (84a)$$

$$\dot{x}_2 = -u_1x_2 + k_0(\tilde{x}_3)\tilde{x}_3(x_1 - x_2) \quad (84b)$$

$$\dot{\tilde{x}}_3 = -\tilde{x}_3\frac{[\ln(\tilde{x}_3/k_{\rm n})]^2}{E}\left[u_1\left(\nu_0 - \frac{E}{\ln(\tilde{x}_3/k_{\rm n})}\right) + C_1\left[x_4 - \frac{E}{\ln(\tilde{x}_3/k_{\rm n})}\right) - \frac{1}{\rho C_\rho}f_{\tilde{x}_3}\right] \quad (84c)$$

$$\dot{x}_4 = \frac{1}{m}u_2 + \frac{C_2}{m}\left[\frac{E}{\ln(\tilde{x}_3/k_{\rm n})} - x_4\right] \quad (84d)$$

where

$$k_0(\tilde{x}_3) = \frac{k_{10}}{k_{\rm n}}{\rm e}^{(E_1-E)/x_3} = \frac{k_{10}}{k_{\rm n}}\left(\frac{\tilde{x}_3}{k_{\rm n}}\right)^{(E_1-E)/E} \quad (84e)$$

Because $E_1 < E$, $k_0(\tilde{x}_3)$ is bounded for all $x_3 \ge 0$. Please note the new definitions for the variables $k_1(\tilde{x}_3)$, $k_3(\tilde{x}_3)$, and $k_0(\tilde{x}_3)$ by comparing the above set of equations with the original system in eqs 4. In the following steps, we propose a Lyapunov function (V_i) at each stage whose derivative is of the form

$$\dot{V}_i = -\alpha V_i + \beta$$

for a particular set of true and virtual inputs. α and β are some positive constants, thus proving that the controller designed in this section is robustly stable. Also note that it is possible to make the control error as small as desired by an appropriate choice of tuning parameters.

Step 1. Consider the following input u_1

$$u_1 = \frac{-c_1z_1 + \dot{x}_{1\rm r} + u_{\rm R1}}{c_{\rm A0} - x_1} \quad (85a)$$

$$u_{\rm R1} = \begin{cases} -[\hat{k}_1(t)^{\rm T}\,\tilde{x}_1(t)]^2z_1(t)/\epsilon_1 & \text{if } |\hat{k}_1\tilde{x}_1z_1| \le \epsilon_1 \\ -\hat{k}_1^{\,\rm T}\,\tilde{x}_1(t){\rm sign}(z_1) & \text{if } |\hat{k}_1\tilde{x}_1z_1| > \epsilon_1 \end{cases} \quad (85b)$$

where $z_1 = x_1 - x_{1\rm r}$, $\tilde{x}_1^{\rm T} = [x_1\ x_1^2]$, $\hat{k}_1 = [\hat{k}_{1\max}\ \hat{k}_{3\max}]$, and $\hat{k}_1$ is updated by

$$\dot{\hat{k}}_1(t) = -\tilde{x}_1|z_1| - \sigma_1\hat{k}_1(t) \quad (86)$$

where $\Gamma = \Gamma^{\rm T}$ is a positive definite matrix and $\sigma_1 > 0$ is scalar tuning parameter. $\hat{k}_{1\max}$ and $\hat{k}_{3\max}$ are the maximum estimated values of $k_1(\tilde{x}_3)$ and $k_3(\tilde{x}_3)$, respectively.

From the assumption that $x_3 > 0$, we have

$$k_1(x_3)x_1 + k_3(x_3)x_1^{\,2} \le k_{1\max}x_1 + k_{3\max}x_1^{\,2} = k_{1\max}^{\ \ \rm T}\tilde{x}_1 \quad (87)$$

Now consider the following positive definite function $\bar{V}_0$

$$\bar{V}_0 = \frac{1}{2}z_1^{\,2} + \frac{1}{2}(\hat{k}_1 - k_{1\max})^{\rm T}\Gamma^{-1}(\hat{k}_1 - k_{1\max}) \quad (88)$$

The time derivative of $\bar{V}_0$, if $|\hat{k}_1^{\rm T}\tilde{x}_1z_1| > \epsilon_1$, is then

$$\begin{aligned}\dot{\bar{V}}_0 &= z_1\dot{z}_1 + (\hat{k}_1 - k_{1\max})^{\rm T}\Gamma^{-1}\dot{\hat{k}}_1(t) \quad (89)\\ &\le -c_1z_1^{\,2} - \sigma_1\lambda_{\min}(\Gamma^{-1})||\hat{k}_1 - k_{1\max}||^2 + \sigma_1||\Gamma^{-1}||\ ||\hat{k}_1 - k_{1\max}||\ ||k_{1\max}|| \\ &\underline{\underline{\Delta}}\ \bar{V}_{01}\end{aligned}$$

In the case where $|\hat{k}_1^{\rm T}\tilde{x}_1z_1| \le \epsilon_1$, we have

$$\dot{\bar{V}}_0 \le \bar{V}_{01} + \epsilon_1$$

It is easy to see that this will ensure the robust stability of the process, i.e., all of the input u_1 and the state x_1 remain bounded. $\bar{V}_{01}$ can be rewritten as

$$\begin{aligned}\bar{V}_{01} &= -c_1z_1^{\,2} - \sigma_1\lambda_{\min}(\Gamma^{-1})||\hat{k}_1 - k_{1\max}||^2 + \sigma_1||\Gamma^{-1}||\ ||\hat{k}_1 - k_{1\max}||\ ||k_{1\max}|| \quad (90)\\ &= -c_1z_1^{\,2} - [\sigma_1\lambda_{\min}(\Gamma^{-1}) - \zeta_1]||\hat{k}_1 - k_{1\max}||^2 - \zeta_1||\hat{k}_1 - k_{1\max}||^2 + \sigma_1||\Gamma^{-1}||\ ||k_{1\max}||\ ||\hat{k}_1 - k_{1\max}||^2 \\ &\le -c_1z_1^{\,2} - (\sigma_1\lambda_{\min}(\Gamma^{-1}) - \zeta_1)||\hat{k}_1 - k_{1\max}||^2 + \frac{(\sigma_1||\Gamma^{-1}||\ ||k_{1\max}||)^2}{4\zeta_1}\end{aligned}$$

By choosing ζ_1 as

$$\zeta_1 = \frac{1}{2}\sigma_1\lambda_{\min}(\Gamma^{-1})$$

we have

$$\dot{\bar{V}}_0 \le -\alpha\bar{V}_0 + \beta \tag{91}$$

where

$$\alpha = \min\left[2c_1, \frac{\sigma_1\lambda_{\min}(\Gamma^{-1})}{\lambda_{\max}(\Gamma^{-1})}\right] \tag{92}$$

$$\beta = \frac{\sigma_1(\|\Gamma^{-1}\|\ \|k_{1\max}\|)^2}{2\lambda_{\min}(\Gamma^{-1})} + \epsilon_1$$

From the above inequality and the fact that $z_1^2 \le 2\bar{V}_0$, it follows that

$$\lim_{t\to\infty}|z_1|^2 \le 2\frac{\beta}{\alpha}$$

Thus, for any δ such that $\delta^2 \ge \beta/\alpha$, we can achieve

$$\lim_{t\to\infty}|z_1| \le \delta \tag{93}$$

It is also clear that the appropriate choice of Γ and ϵ_1 ensures the inequality. For instance, it is sufficient to set

$$\Gamma = \gamma I_2 \tag{94}$$

$$\gamma > \frac{2\sigma_1\|k_{\max}\|^2}{\alpha\delta^2}$$

$$\epsilon_2 < \frac{\alpha\delta^2}{4}$$

where $\alpha = \min(2c_1, \delta_1)$. It can similarly be proved that z_2, z_3, and z_4 can be reduced to a desired level by appropriately choosing the tuning parameters.

Step 2. Define $z_2 = x_2 - x_{2r}$. It then follows that

$$\dot{z}_2 = -u_1x_2 + k_0(\tilde{x}_3)\tilde{x}_3(x_1 - x_2) - \dot{x}_{2r} \tag{95}$$

Choose

$$\tilde{x}_{3r} = \frac{-c_2z_2 + u_1x_2 + \dot{x}_{2r}}{\hat{k}_{0\max}(x_1 - x_2)} + u_{R2} \tag{96}$$

where $\hat{k}_{0\max}$ is the estimated maximum value of $k_0(\tilde{x}_3)$ and u_{R2} is a robust control term. γ_2, c_2, and σ_2 are positive tuning parameters. The update law for $\hat{k}_{0\max}$ and the robust control signal are designed as follows

$$\dot{\hat{k}}_{0\max} = \gamma_2\frac{|-c_2z_2 + u_1x_2 + \dot{x}_{2r}|}{\hat{k}_{0\max}}|z_2| - \sigma_2\hat{k}_{0\max} \tag{97}$$

with the initial condition $\hat{k}_{\max}(0) > 0$

$$u_{R2} = \begin{cases} 0 & \text{if } (-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 \ge 0 \\ -\dfrac{2(-c_2z_2 + u_1x_2 + \dot{x}_{2r})^2}{\epsilon_2k_{0m}(x_3)(x_1 - x_2)} & \text{if } -\epsilon_2 \ge (-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 < 0 \\ \dfrac{2(-c_2z_2 + u_1x_2 + \dot{x}_{2r})}{k_{0m}(x_3)(x_1 - x_2)} & \text{if } (-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 < -\epsilon_2 \end{cases} \tag{98}$$

for $\epsilon 2 > 0$, where

$$k_{0m}(x_3) = \frac{k_{0m}}{k_n}e^{(E_2 - E)/x_3} < k_0(\tilde{x}_3)$$

k_{0m} is a lower bound of k_{10}. Considering a positive definite function

$$\bar{V}_1 = \frac{1}{2}{z_2}^2 + \frac{1}{2\gamma_2}(\hat{k}_{0\max} - k_{0\max})^2 \tag{99}$$

we have

$$\begin{aligned}\dot{\bar{V}}_1 = {} & (-u_1x_2 - \dot{x}_{2r})z_2 + k_0(\tilde{x}_3)z_3(x_1 - x_2)z_2 + k_0(\tilde{x}_3)u_{R2}(x_1 - x_2)z_2 \\ & + \frac{k_0(\tilde{x}_3)}{\hat{k}_{0\max}}(-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 + \frac{1}{\gamma_1}(\hat{k}_{0\max} - k_{0\max})\dot{\hat{k}}_{0\max} \\ \le {} & -c_2{z_2}^2 + k_0(\tilde{x}_3)(x_1 - x_2)z_2z_3 + k_0(\tilde{x}_3)u_{R2}(x_1 - x_2)z_2 \\ & + \frac{k_0(\tilde{x}_3)}{\hat{k}_{0\max}}(-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 - (-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 \\ & + |-c_2z_2 + u_1x_2 + \dot{x}_{2r}||z_2| - \frac{k_{0\max}}{\hat{k}_{0\max}}|-c_2z_2 + u_1x_2 + \dot{x}_{2r}||z_2| \\ & - \frac{\sigma_2}{\gamma_2}(\hat{k}_{0\max} - k_{0\max})^2 + \frac{\sigma^2}{\gamma_2}|\hat{k}_{0\max} - k_{0\max}|k_{0\max}\end{aligned} \tag{100}$$

If $(-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 \ge 0$, then we have

$$\begin{aligned}\dot{\bar{V}}_1 \le {} & -c_2{z_2}^2 + k_0(\tilde{x}_3)(x_1 - x_2)z_2z_3 - \frac{\sigma_2}{\gamma_2}(\hat{k}_{0\max} - k_{0\max})^2 + \frac{\sigma_2}{\gamma_2}|\hat{k}_{0\max} - k_{0\max}|k_{0\max} \\ \stackrel{\Delta}{=} {} & \bar{V}_{11} + k_0(\tilde{x}_3)(x_1 - x_2)z_2z_3\end{aligned} \tag{101}$$

If $(-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 < -\epsilon_2$, then

$$\begin{aligned}\dot{\bar{V}}_1 \le {} & \bar{V}_{11} + \frac{k_0(\tilde{x}_3)}{k_{0m}(x_3)}2(-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 + 2|-c_2z_2 + u_1x_2 + \dot{x}_{2r}||z_2| + k_0(\tilde{x}_3)(x_1 - x_2)z_2z_3 \\ \le {} & \bar{V}_{11} + k_0(\tilde{x}_3)(x_1 - x_2)z_2z_3\end{aligned} \tag{102}$$

In the case where $\epsilon_2 \le (-c_2z_2 + u_1x_2 + \dot{x}_{2r})z_2 < 0$, we have

$$\dot{\bar{V}}_1 \le \bar{V}_{11} + k_0(\tilde{x}_3)(x_1 - x_2)z_2z_3 + 2\epsilon_2 \quad (103)$$

Step 3. Defining

$$z_3 = \tilde{x}_3 - \tilde{x}_{3r}$$

we obtain

$$\frac{1}{C_1}\dot{z}_3 = -\tilde{x}_3\frac{[\ln(\tilde{x}_3/k_n)]^2}{E}x_4 - \tilde{x}_3\frac{[\ln(\tilde{x}_3/k_n)]^2}{E}\left[\frac{1}{C_1}u_1\left(\nu_0 - \frac{E}{\ln(\tilde{x}_3/k_N)}\right) - \frac{E}{\ln(\tilde{x}_3/k_n)} - \frac{1}{\rho C_1C_p}f_x\right] - \frac{1}{C_1}\dot{x}_{3r} \quad (104)$$

where

$$\frac{1}{\rho C_1C_p}f_x = \frac{1}{\rho C_1C_p}[k_1(x_3)x_1d_1 + k_2(x_3)x_2d_2 + k_3(x_3)x_1^2d_3] \quad (105)$$

$$\le k_{d_1\max}x_1 + k_{d_2\max}x_2 + k_{d_3\max}x_1^2$$

$$= k^T d_{\max}\tilde{x}_f$$

$$\tilde{x}_f = [x_1\ x_2\ x_1^2]^T$$

Choosing x_{4r} as

$$x_{4r} = \frac{E}{\ln(\tilde{x}_3/k_n)} + \frac{c_3E}{\tilde{x}_3[\ln(\tilde{x}_3/k_n)]^2}z_3 + u_{R3} + u_{R4} - \frac{E}{\tilde{x}_3[\ln(\tilde{x}_3/k_n)]^2}u_{R5} - \frac{E}{\tilde{x}_3[\ln(\tilde{x}_3/k_n)]^2} \quad (106)$$

where

$$u_{R3} = \hat{\beta}_1\left[u_1\left(\nu_0 - \frac{E}{\ln(\tilde{x}_3/k_n)}\right) + \frac{E}{\tilde{x}_3[\ln(\tilde{x}_3/k_n)]^2}\dot{x}_{3r}\right] \quad (107)$$

$$\dot{\hat{\beta}}_1 = \gamma_3\frac{-\tilde{x}_3[\ln(\tilde{x}_3/k_n)]^2}{E}\left[u_1\left(\nu_0 - \frac{E}{\ln(\tilde{x}_3/k_n)}\right) + \frac{E}{\tilde{x}_3[\ln(\tilde{x}_3/k_n)]^2}\dot{x}_{3r}\right]z_3 - \sigma_3\hat{\beta}_1 \quad (108)$$

$$\gamma_3 > 0, \quad \sigma_3 > 0 \quad (109)$$

$$u_{R4} = \begin{cases} -\left[\hat{k}_d^T\tilde{x}_f\left|\frac{\tilde{x}_3\ln(\tilde{x}_3/k_n)^2}{E}\right|\right]^2 z_3/\epsilon_4 & \text{if } |\hat{k}_d^T\tilde{x}_f\left|\frac{\tilde{x}_3\ln(\tilde{x}_3/k_n)^2}{E}\right|z_3| \le \epsilon 4 \\ -\hat{k}_d^T\tilde{x}_f\left|\frac{\tilde{x}_3\ln(\tilde{x}_3/k_n)^2}{E}\right|\text{sign}(z_3) & \text{if } |\hat{k}_d^T\tilde{x}_f\left|\frac{\tilde{x}_3\ln(\tilde{x}_3/k_n)^2}{E}\right|z_3| > \epsilon 4 \end{cases} \quad (110)$$

$$\dot{\hat{k}}_d = \Gamma_d\tilde{x}_f\left|\frac{\tilde{x}_3\ln(\tilde{x}_3/k_n)^2}{E}\right||z_3| - \sigma_d\hat{k}_d \quad (111)$$

$$\Gamma_d = \Gamma_d^T > 0, \quad \sigma_d > 0$$

$$u_{R5} = \begin{cases} -[\hat{k}_5|z_2(x_1 - x_2)|]^2z_3/\epsilon_5 & \text{if } |z_2(x_1 - x_2)z_3| \le \epsilon_5 \\ \hat{k}_4|z_2(x_1 - x_2)|\text{sign}(z_3) & \text{if } |z_2(x_1 - x_2)z_3| > \epsilon_5 \end{cases} \quad (112)$$

$$\dot{\hat{k}}_5 = \gamma_5|z_2(x_1 - x_2)||z_3| - \sigma_5\hat{k}_5 \quad (113)$$

$$\sigma_5 > 0, \quad \gamma_5 > 0$$

By considering the following positive definite function

$$\bar{V}_2 = \bar{V}_1 + \frac{1}{2C_1}z_3^2 + \frac{1}{2\gamma_3}\left(\hat{\beta}_1 - \frac{1}{C_1}\right)^2 + \frac{1}{2}(\hat{k}_d - k_{d\max})^T\gamma_d^{-1}(\hat{k}_d - k_{d\max}) + \frac{1}{2\gamma_5}(\hat{k}_5 - k_{0\max})^2 \quad (114)$$

we have

$$\dot{\bar{V}}_2 \le \bar{V}_{11} + 2\epsilon_f - \tilde{x}_3\frac{[\ln(\tilde{x}_3/k_n)]^2}{E}z_3z_4 - c_3z_3^2 + k_{0\max}|z_2(x_1 - x_2)||z_3| \quad (115)$$

$$+ \tilde{x}_3\frac{[\ln(\tilde{x}_3/k_n)]^2}{E}\left(\hat{\beta}_1 - \frac{1}{C_1}\right)\left[u_1\left(\nu_0 - \frac{E}{\ln(\tilde{x}_3/k_n)}\right) + \frac{E}{\tilde{x}_3[\ln(\tilde{x}_3/k_n)]^2}\dot{x}_{3r}\right]z_3$$

$$+ k_{d\max}^T\tilde{x}_f\left|\tilde{x}_3\frac{[\ln(\tilde{x}_3/k_n)]^2}{E}\right||z_3|u_{R4} + u_{R5}$$

$$- \tilde{x}_3\frac{[\ln(\tilde{x}_3/k_n)]^2}{E}\left(\hat{\beta}_1 - \frac{1}{C_1}\right)\left[u_1\left(\nu_0 - \frac{E}{\ln(\tilde{x}_3/k_n)}\right) + \frac{E}{\tilde{x}_3[\ln(\tilde{x}_3/k_n)]^2}\dot{x}_{3r}\right]z_3$$

$$- \frac{\sigma_3}{\gamma_3}\left(\hat{\beta}_1 - \frac{1}{C_1}\right)^2 + \frac{\sigma_3}{\gamma_3}\left|\hat{\beta}_1 - \frac{1}{C_1}\right|\frac{1}{C_1} + (\hat{k}_d - k_{d\max})^T\tilde{x}_f\left|\tilde{x}_3\frac{[\ln(\tilde{x}_3/k_n)]^2}{E}\right||z_3|$$

$$- \sigma_d\lambda_{\min}(\Gamma^{-1})||\hat{k}_d - k_{d\max}||^2 + \sigma_d||\Gamma_d^{-1}||\ ||\hat{k}_d - k_{d\max}||\ ||k_{d\max}||$$

$$+ (\hat{k}_5 - k_{0\max})|z_2(x_1 - x_2)||z_3| - \frac{\sigma_5}{\gamma_5}(\hat{k}_5 - k_{0\max})^2 + \frac{\sigma_5}{\gamma_5}|\hat{k}_5 - k_{0\max}|k_{0\max}$$

Finally, we obtain

$$\dot{\bar{V}}_2 \le \bar{V}_{11} + 2\epsilon_f - \tilde{x}_3\frac{[\ln(\tilde{x}_3/k_n)]^2}{E}z_3z_4 - c_3z_3^2 - \frac{\sigma_3}{\gamma_3}\left(\hat{\beta}_1 - \frac{1}{C_1}\right)^2 + \frac{\sigma_3}{\gamma_3}\left|\hat{\beta}_1 - \frac{1}{C_1}\right|\frac{1}{C_1} \quad (116)$$

$$- \sigma_d\lambda_{\min}(\Gamma^{-1})||\hat{k}_d - k_{d\max}||^2 + \sigma_d||\Gamma^{-1}||\ ||\hat{k}_d - k_{d\max}||\ ||k_{d\max}||$$

$$- \frac{\sigma_5}{\gamma_5}(\hat{k}_5 - k_{0\max})^2 + \frac{\sigma_5}{\gamma_5}|\hat{k}_5 - k_{0\max}|k_{0\max} + \epsilon_4 + \epsilon_5$$

$$= \bar{V}_{11} + 2\epsilon_f + \bar{V}_{21} + \epsilon_4 + \epsilon_5 - \tilde{x}_3\frac{[\ln(\tilde{x}_3)]^2}{E}z_3z_4$$

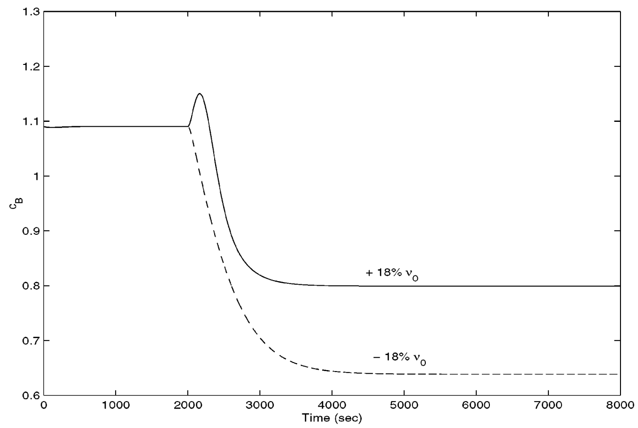

Figure 2. Open-loop behavior of the yield of product B under changes in temperature ν_0.

where

$$\bar{V}_{21} = -c_3 z_3^{\,2} - \frac{\sigma_3}{\gamma_3}\left(\hat{\beta}_1 - \frac{1}{C_1}\right)^2 + \frac{\sigma_3}{\gamma_3}\left|\hat{\beta}_1 - \frac{1}{C_1}\right|\frac{1}{C_1} - \sigma_d \lambda_{\min}(\Gamma^{-1})||\hat{k}_d - k_{dmax}||^2 + \sigma_d||\Gamma^{-1}||\,||\hat{k}_d - k_{dmax}||\,||k_{dmax}|| - \frac{\sigma_5}{\gamma_5}(\hat{k}_5 - k_{0max})^2 + \frac{\sigma_5}{\gamma_5}|\hat{k}_5 - k_{0max}|k_{0max} \quad (117)$$

Step 4. The fourth error system is

$$z_4 = x_4 - x_{4r} \quad (118)$$

$$m\dot{z}_4 = u_2 + C_2\left[\frac{E}{\ln(\tilde{x}_3/k_n)} - x_4\right] - m\dot{x}_{4r}$$

We design u_2 as

$$u_2 = -c_2 z_4 + \tilde{x}_3\frac{[\ln(\tilde{x}_3/k_n)]^2}{E}z_3 - \hat{C}_2\left[\frac{E}{\ln(\tilde{x}_3/k_n)} - x_4\right] + \hat{m}\dot{x}_{4r} \quad (119)$$

where

$$\dot{\hat{C}}_2 = \gamma_6\left(\frac{E}{\ln(\tilde{x}_3/k_n)} - x_4\right)z_4 - \sigma_6\hat{C}_2 \quad (120)$$

$$\dot{\hat{m}} = -\gamma_7\,\dot{x}_{4r}z_4 - \sigma_4\hat{m} \quad (121)$$

$$\gamma_6 > 0, \quad \sigma_6 > 0, \quad \gamma_7 > 0, \quad \sigma_7 > 0$$

Setting a final positive definite function as

$$\bar{V}_3 = \bar{V}_2 + \frac{1}{2}mz_4^{\,2} + \frac{1}{2_{\gamma 6}}(\hat{C}_2 - C_2)^2 + \frac{1}{2\gamma_7}(\hat{m} - m)^2 \quad (122)$$

we have

$$\dot{\bar{V}}_3 \le \bar{V}_{11} + \bar{V}_{21} + 2\epsilon_f + \epsilon_4 + \epsilon_5 - c_4 z_4^{\,2} - \frac{\sigma_6}{\gamma_6}(\hat{C}_2 - C_2)^2 + \frac{\sigma_6}{\gamma_6}|\hat{C}_2 - C_2|C_2 - \frac{\sigma_7}{\gamma_7}(\hat{m} - m)^2 + \frac{\sigma_7}{\gamma_7}|\hat{m} - m|m \quad (123)$$

From the eqs 103, 117, and 122, one can easily show that all of the signals in the control system are bounded so that the process is robustly stable. A method similar to that applied in step 1 can be used to prove that the errors converge to a value less than the specified error bound. The key to achieving the desired error bounds lies in choosing appropriate values of γ_i, σ_i, and ϵ_i.

6. Simulations and Results

The particular CSTR model used in this project, as well as in many other studies, is highly nonlinear and

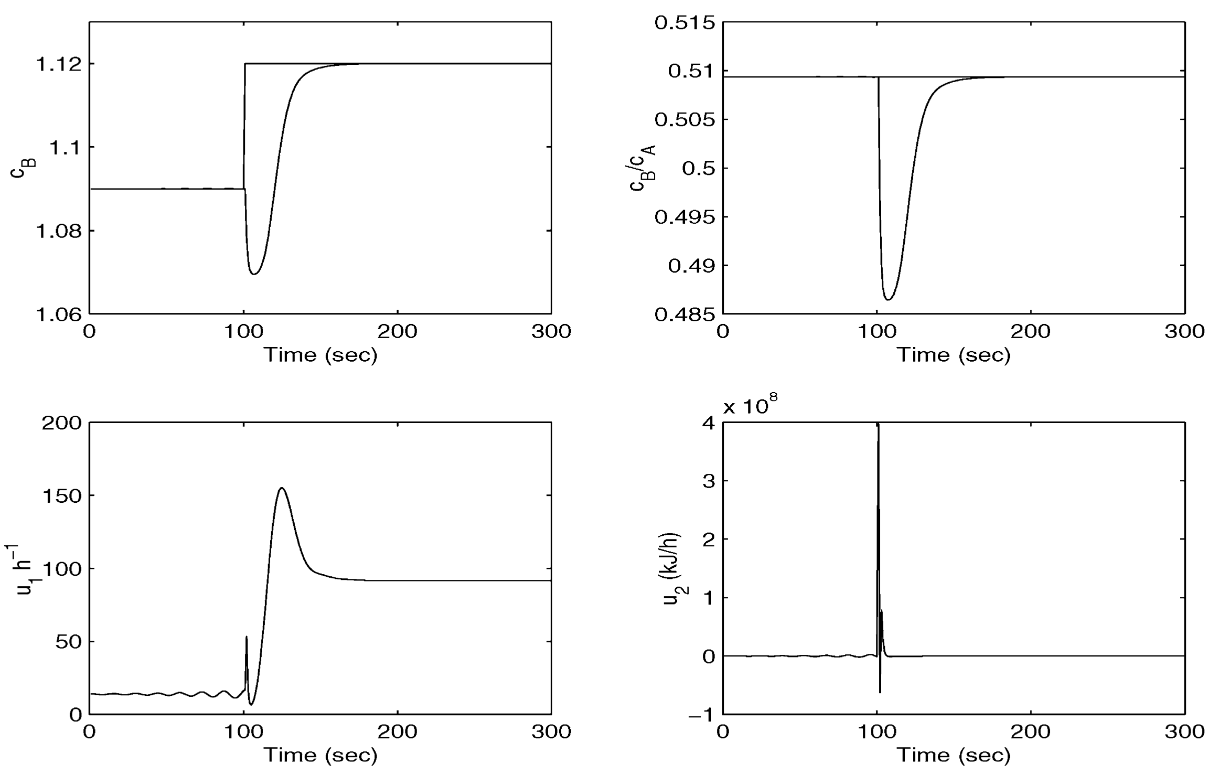

Figure 3. Set-point tracking and input magnitudes.

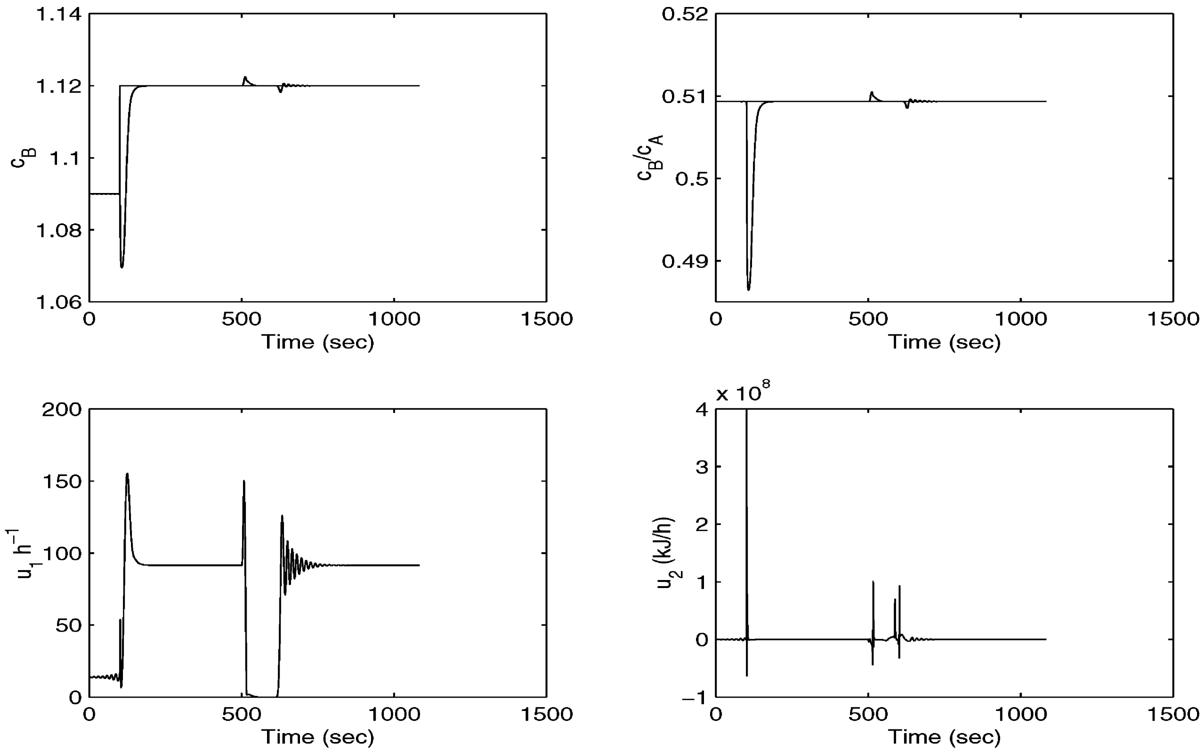

Figure 4. Disturbance rejection.

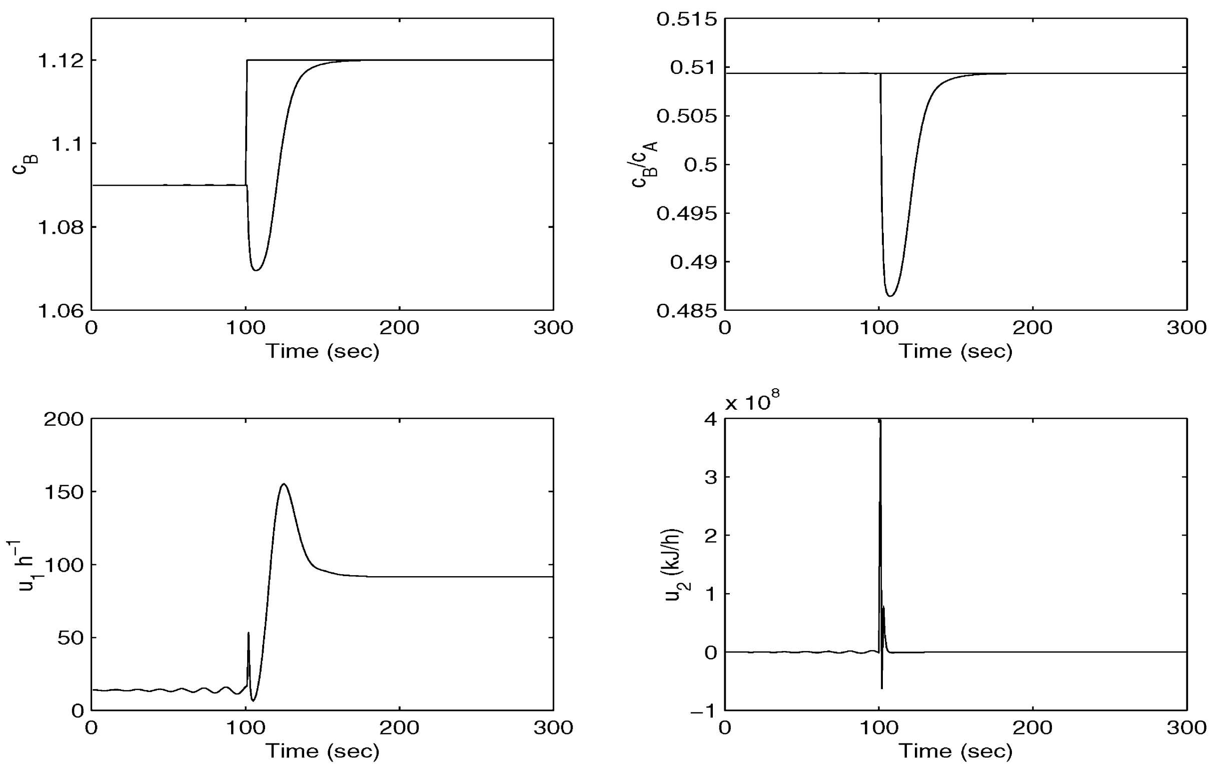

Figure 5. Controller without cancellation useful nonlinearities.

very difficult to control. In ref 17, an optimum steady state of the yield Φ of product B with respect to the design variables was obtained. (Note that Figure 1 also gives the same optimum c_A and c_B) They found the following optimal operating point

$$c_{A0}|_s = 5.10 \text{ mol/L}, \qquad c_A|_s = 2.14 \text{ mol/L}$$

$$\nu_0|_s = 104.9 \text{ °C}, \qquad c_B|_s = 1.09 \text{ mol/L}$$

$$\left.\frac{\dot{V}}{V_R}\right|_s = 14.9 \text{ h}^{-1}, \qquad \nu|_s = 114.2 \text{ °C}$$

$$\dot{Q}_K|_s = -1113.5 \text{ kJ/h}, \quad \nu_K|_s = 112.9 \text{ °C}$$

However, under real operating conditions, the operating point must quite often be changed. Moreover, the presence of disturbances makes it very important to

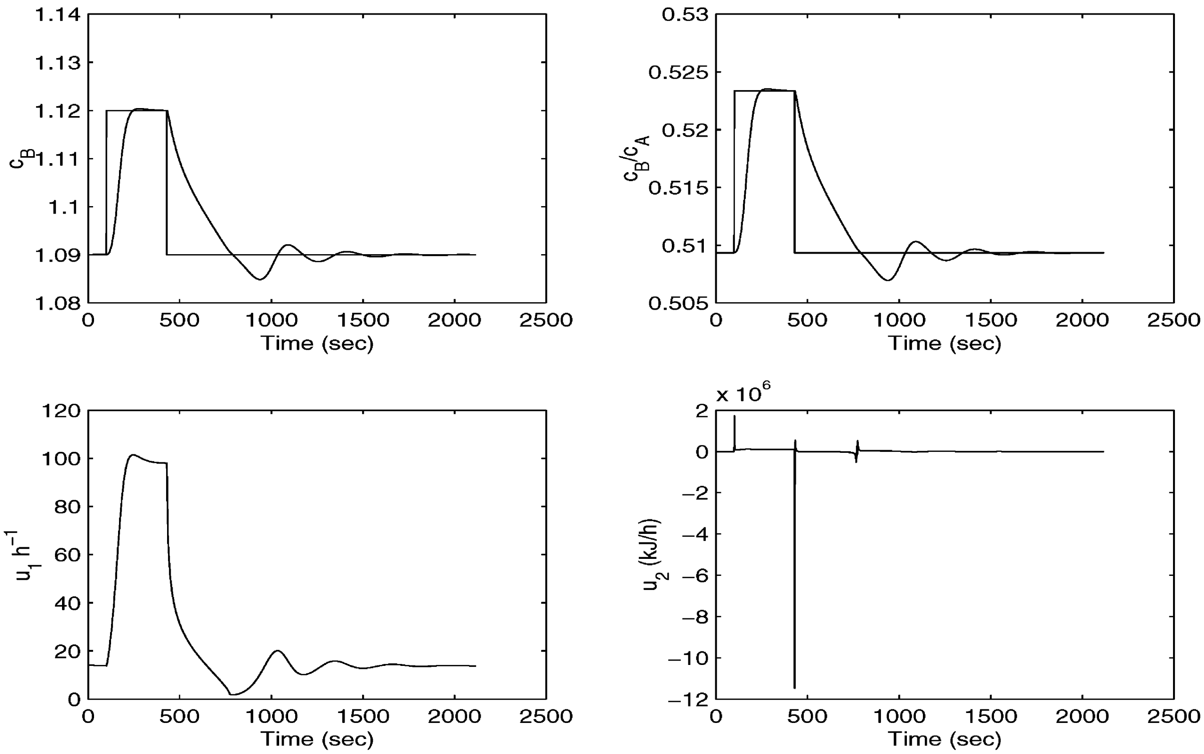

Figure 6. Tracking in the absence of derivatives.

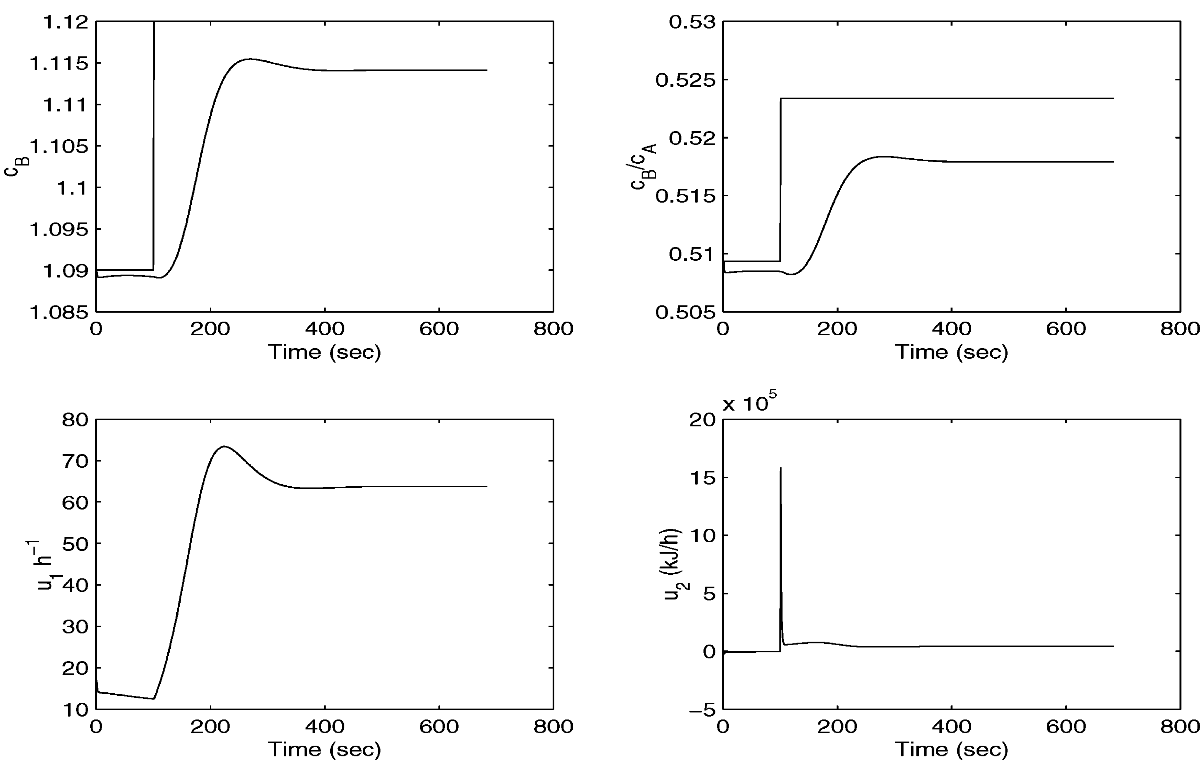

Figure 7. Tracking with a nonadaptive controller in the presence of uncertain parameters.

regulate the outputs. The open-loop behavior of the CSTR model under increments of +18 and −18% in the steady-state value of ν_0 is shown in Figure 2. The controller performance for $c_1 = 1$, $c_2 = 2$, $c_3 = 0.1$, and $c_4 = 1$ is shown in Figure 3. It was found that the transient performance of the controller can be improved by an appropriate choice of c_i's, but it is difficult to define and find what is appropriate. The above choice of c_i's was found to give reasonable performance (it was obtained by trial and error). The performance of the controller under disturbances is shown in Figure 4. A +18% increase in ν_0 was introduced at $t = 500$ s. Figure 7 shows the performance of a nonadaptive backstepping controller that uses estimated parameters instead of the

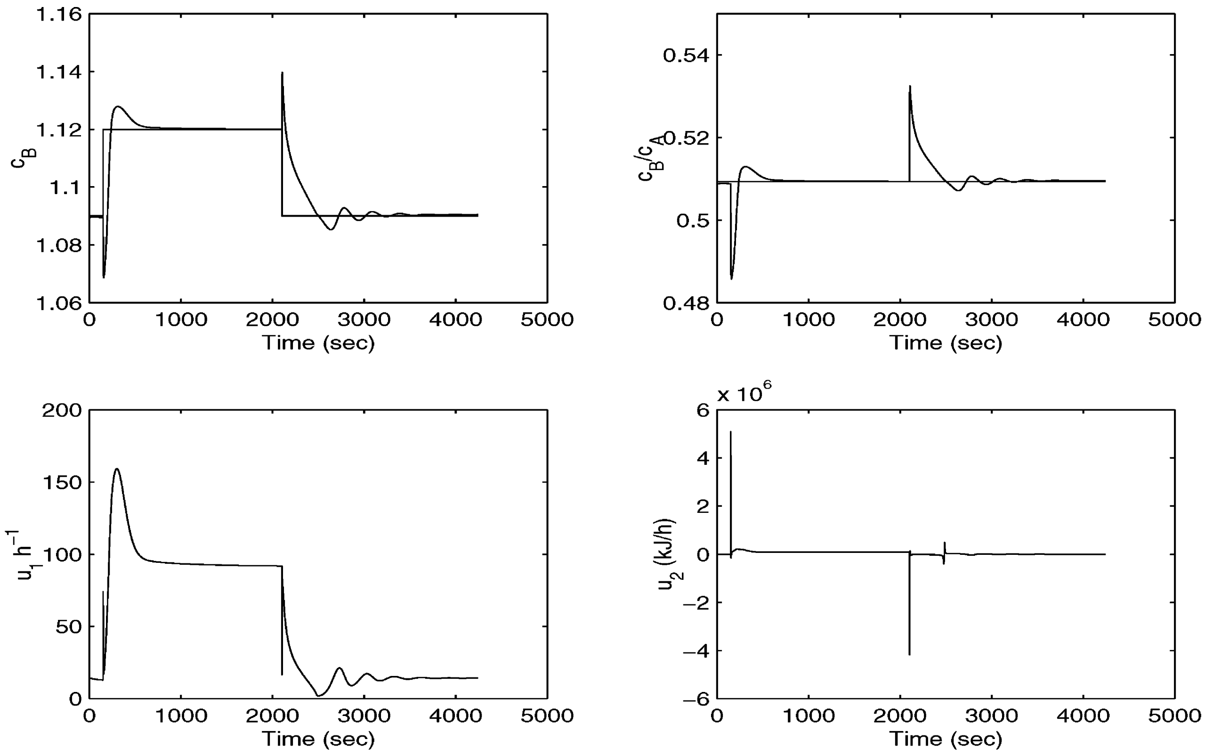

Figure 8. Adaptive backstepping.

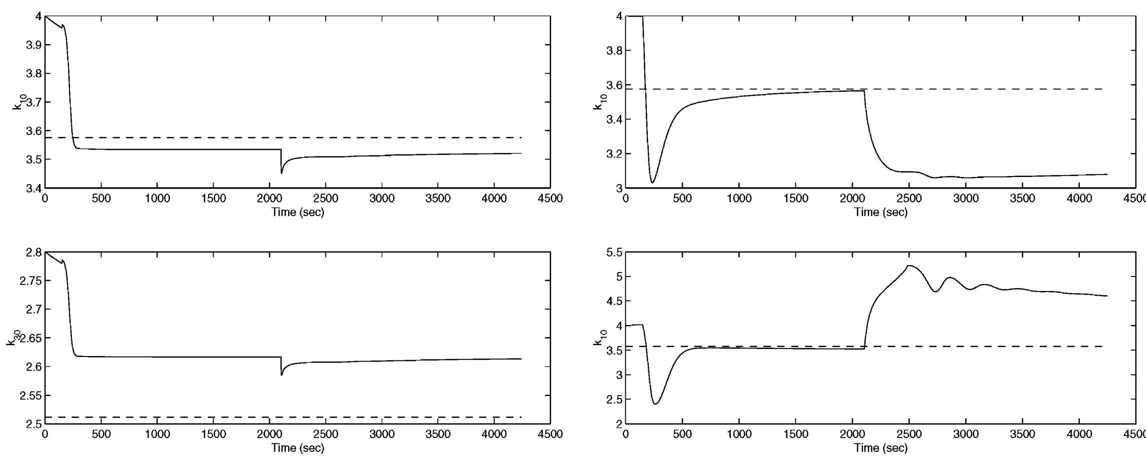

Figure 9. Uncertain parameter trajectories: (a) from the first two update laws, (b) from the third and fourth update laws.

true parameters. The following estimated values of the uncertain parameters were used

$$k_{10} = 4 \times 10^8$$

$$k_{20} = 4 \times 10^8$$

$$k_{30} = 2.8 \times 10^6$$

Clearly, if the estimated values are not the true values, a nonadaptive backstepping controller cannot track the set point. The performance of the adaptive controller with update laws for the uncertain parameters is shown in Figure 8. The initial values of the estimated parameters are the same as above. The uncertain parameters from the five update laws obtained from the original method are shown in Figures 9 and 10. The controller response from the modified method and the tuning function update laws are shown in Figures 11 and 12, respectively. A disturbance of magnitude +18% of steady-state value of v_0 is intro-

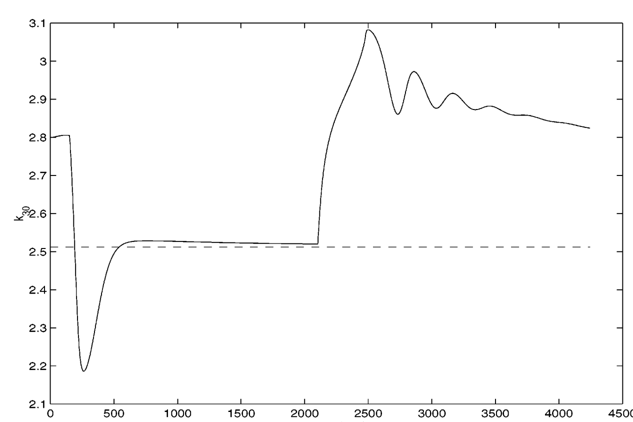

Figure 10. Uncertain parameters from the fifth update law.

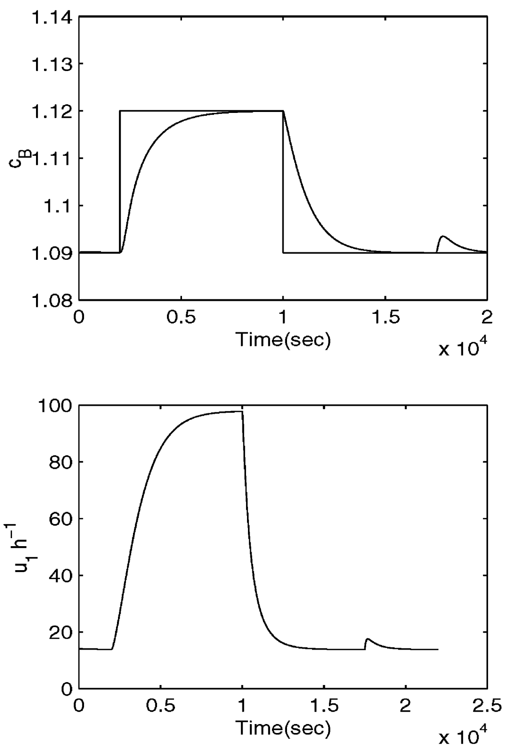

Figure 11. Step tracking with the adaptive controller.

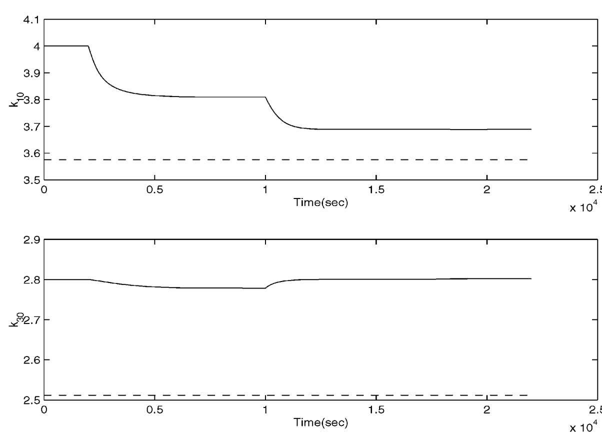

Figure 12. Update laws in eqs 82 and 83.

duced at $t = 17\,500$ s. Observe that both inputs in this figure are within achievable levels. This is made possible by an appropriate choice of tuning parameters (c_i's). In this case, however, the time required to reach the set point is very large. Thus, there is a trade off between the performance of the controller and the input energy spent. Finally, the results obtained by the robust adaptive controller are shown in Figure 13. The Tables 2 and 3 lists the various tuning parameters used in the simulations.

Table 2. Various Tuning Parameters Used in Simulations

figure(s)	c_1	c_2	c_3	c_4	γ_1	γ_2	γ_3	γ_4	γ_5
3−5	1	2	0.1	1	−	−	−	−	−
7	1	1	0.001	1	−	−	−	−	−
8−10	1	1	0.001	1	10	10	5	0.1	0.05
11, 12	0.001	0.1	0.001	0.01	0.1	0.1	−	−	−

Table 3. Tuning Parameters Used in the Robust Adaptive Controller

tuning parameter	value	tuning parameter	value
Γ	[1 0; 0 1]	σ_1	1×10^{-8}
γ_2	1×10^{-6}	σ_2	1×10^{-8}
γ_3	0.1	σ_3	0.01
Γ_d	diag(1×10^{-2},	σ_d	1×10^{-4}
	1×10^{-3},	σ_5	1×10^{-6}
	1×10^{-2})[a]	σ_6	0.01
γ_5	0.1	σ_7	0.01
γ_6	0.001	c_2	30
γ_7	0.001	c_4	15
c_1	6	k_{10}	3.575×10^8
c_3	0.1	$\hat{m}(0)$	0
E	−9500	e_2	0.1
k_n	3.575×10^5	e_5	0.004
e_1	0.01	$\hat{k}_{0\max}(0)$	$1 \times 10^2 e^{(E1-E)/387.2}$
e_4	0.1	$\hat{k}_d(0)$	$[0.01\ 0.01\ 0.01]^T$
$\hat{k}(0)$[b]	[0; 0]	$\hat{C}_2(0)$	0
$\hat{\beta}(0)$	$0.1/C_1$		
$\hat{k}_5(0)$	$10e^{(E1-E)/387.2}$		
E_{02}	$1.1E$		

[a] diag(x) stands for a diagonal matrix with elements of vector x along the diagonal. [b] The zero in parentheses denotes the initial value of this parameter.

7. Conclusions

We have applied the technique of backstepping to design a nonlinear controller for a benchmark CSTR and studied its performance. We also designed an adaptive backstepping controller that can track signals in the presence of large uncertainties in the estimated plant. Tuning functions were designed to reduce the number of update laws. Finally, a robust nonlinear adaptive controller was derived using Lyapunov stability theory. Although the backstepping approach is simple and efficient, it has a few drawbacks. A backstepping controller cannot guarantee asymptotic tracking in the presence of state or output noise and/or input constraints. We believe input constraints can be incorporated into this scheme, which we will investigate as part of our future work.

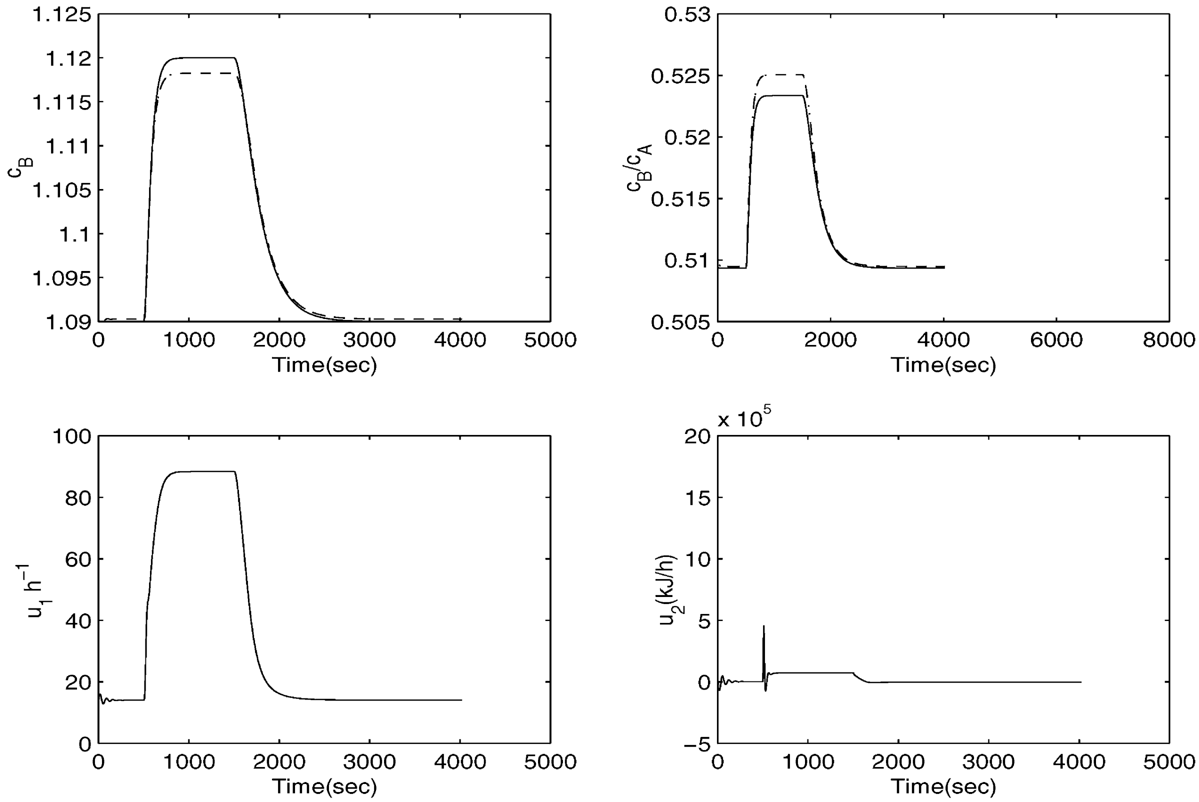

Figure 13. Response of the robust adaptive controller. The thick line is the set point, and the dashed line is the CSTR output.

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