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SHORT PAPER

L_1 Control for Positive Markovian Jump Systems with Time-Varying Delays and Partly Known Transition Rates

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Abstract This paper deals with the problem of L_1 control for positive Markovian jump systems with time-varying delays and partly known transition rates. Firstly, by the use of appropriate co-positive type Lyapunov function, sufficient conditions for stochastic stability of positive Markovian jump systems with time-varying delays and partly known transition rates are proposed. Then, L_1-gain performance of the system considered is analyzed. Based on the results obtained, a state feedback controller is constructed such that the closed-loop Markovian jump system is positive and stochastically stable with L_1-gain performance. All the proposed conditions are derived in linear programming. Finally, an example is given to demonstrate the validity of the main results.

Keywords Positive Markovian jump systems · Partly known transition rates · Stochastic stability · Linear programming

1 Introduction

A positive system means that its state and output are nonnegative whenever initial conditions and inputs are nonnegative. There are many applications of such systems in practice such as communication networks [15], industrial engineering [2], and system control theory [6]. As a consequence, the study of positive systems has become a heated topic due to their broad applications in the control community such as stability

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[3,5,7,25,28,29], stabilization [3,7], and H_∞ filtering [10]. To mention a few, by applying appropriate co-positive type Lyapunov function, sufficient conditions for stability criteria of continuous-time or discrete-time positive switched systems [3,28, 29] were proposed in linear programming.

On the other hand, it is widely known that the reaction of real-world systems to exogenous signals is always not instantaneous and is affected by certain time delays, such as networked control systems and transportation systems. Time delay is frequently a source of instability and undesirable performance. Recently, studies of positive time-delayed systems have attracted more and more attention such as stability and stabilization analysis [12,18,31], fault detection [19], and finite-time control [11,17].

Furthermore, dynamic systems subject to random abrupt variations can be modeled as Markovian jump systems such as networked control systems [4], manufacturing systems [14], and fault-detection systems [8]. As a main factor, the transition probability determines the behavior of Markovian jump system. Until now, most of the results based on Markovian jump system have been reported under the assumption of a completely accessible knowledge of the transition probabilities [9,13]. However, the knowledge on the transition probabilities cannot be completely available due to complex factors. Consequently, some achievements on the Markovian jump systems with partly known transition probabilities have been conducted including stability and stabilization analysis [21,24], H_∞ performance analysis [27], filter design [22], actuator saturation [16], finite-time control [26], and fault-detection systems [20].

Recently, there have been a limited amount of papers reporting on the positive Markovian jump system [1,23,30]. To mention a few, sufficient and necessary conditions for stochastic stability of continuous-time or discrete-time positive Markovian jump system [1,23] were proposed in linear programming. And a state feedback controller for continuous-time and discrete-time positive Markovian jump system [23] was designed to ensure positivity and stochastic stability of the closed-loop Markovian jump system. In [30], sufficient and necessary conditions are given so that the discrete-time positive Markovian jump system is stochastically stable with L_1-gain performance. Additionally, the positive L_1 filter design for discrete-time Markovian jump system with partly known transition probabilities was addressed in [30], which is the first paper to consider the discrete-time positive Markovian system with exogenous disturbances and partly known transition probabilities.

At the same time, it is widely recognized that many previous results of the positive systems were based on quadratic Lyapunov functions, and these results were formulated in the terms of linear matrix inequality [5,7,10]. However, some results based on linear co-positive Lyapunov functions have been presented in [1,3,11,17–19,23,28–31]. Due to the nonnegative property of positive systems, a linear Lyapunov function can be chosen as a valid candidate. The consideration of a linear Lyapunov function leads to some new results for positive systems. Compared with previous results based on quadratic Lyapunov functions, the results with linear Lyapunov functions are easy to analyze. Additionally, there are many different performances in the existing literatures such as H_∞ performance [10], H_2 performance, L_2 performance, and so on. When considering the linear Lyapunov function candidate, L_1 performance for positive system was first introduced in [3]. In some situations, these performance measures induced by L_2 signals are appropriate to describe the dynamic behavior. In addition,

L_1-norm provides a more useful description for positive systems because it represents the sum of the values of the components.

For a positive Markovian jump system, it is an area that is only beginning to be investigated, and there is further room for investigation. As pointed out in [1,23], the transition rates are completely known, which may lead to some conservativeness. To the best of our knowledge, most of the existing results about the problems of analysis and synthesis for positive Markovian jump systems are based on nominal system without taking time delay and exogenous disturbance into account. As time delay and exogenous disturbance are frequently encountered in practice, it is necessary and significant to further consider positive Markovian jump system with those two kinds of phenomenons. When taking time delays and exogenous disturbance into account, the problem of choosing an appropriate mode-dependent co-positive Lyapunov function candidate that is different from that in the existing literatures, and as to how to reduce some conservativeness of Lyapunov function are more complicated and challenging. However, until now, no relevant work that considers this kind of system has been published, which motivates our investigation.

In this paper, L_1 control for the positive Markovian jump systems with time-varying delays and partly known transition rates will be investigated. The main contributions of this paper include (i) by employing an appropriate co-positive type Lyapunov function, sufficient conditions for stochastic stability of the system are proposed; (ii) L_1-gain performance analysis for the considered system is analyzed based on the stochastic stability in (i); (iii) based on the results obtained, a state feedback controller is designed to ensure positivity and L_1 boundedness of the closed-loop system.

The remainder of this paper is organized as follows. In Sect. 2, the system formulation and some necessary lemmas are given. Section 3 is devoted to deriving the results on stochastic stability, L_1-gain analysis and controller design. An example is provided to illustrate the feasibility of the obtained results in Sect. 4. Concluding remarks are given in Sect. 5.

Notation In this paper, $A \succeq (\preceq 0, \succ, \prec)$ means that all entries of matrix A are nonnegative (non-positive, positive, negative); $A \succ B\,(A \succeq B)$ means that $A - B \succ 0\,(A - B \succeq 0)$; $R\,(R_+)$ is the set of all real (positive real) numbers; $R^n\,(R^n_+)$ in n-dimensional real (positive) vector space; The vector 1-norm is denoted by $||x||_1 = \sum_{k=1}^{n} |x_k|$, where x_k is the kth element of $x \in R^n$; Given $v : R \rightarrow R^n$, the L_1-norm is defined by $||v||_{L_1} = \int_0^\infty ||v||_1 \mathrm{d}t$; $L_1[0, +\infty)$ is the space of absolute integrable vector-valued functions on $[0, +\infty)$, i.e., we say $x : [0, +\infty) \rightarrow R^n$ is in $L_1[0, +\infty)$ if $\int_0^\infty ||x(t)||_1 \mathrm{d}t < \infty$; Matrix A is said to be a Metzler matrix if its off-diagonal elements are all nonnegative real numbers; Symbol $E\{\cdot\}$ represents the mathematical expectation; I_n denotes identity matrix, and $\mathbf{1}_n$ means the all-ones vector in R^n. Given a probability space $(\varXi, \varUpsilon, \varTheta)$, $\varXi$ is the sample space, $\varUpsilon$ is the σ-algebra of subsets of the sample space, and $\varTheta$ is the probability measure on $\varUpsilon$.

2 Problem Statement and Preliminaries

Consider the following positive Markovian jump system with time-varying delays on the probability space $(\varXi, \varUpsilon, \varTheta)$:

$$\begin{aligned} \dot{x}(t) &= A(g_t)x(t) + A_d(g_t)x(t-\tau(t)) + G(g_t)w(t), \\ z(t) &= C(g_t)x(t) + D(g_t)w(t), \\ x(t+\theta) &= \varphi(\theta), \forall \theta \in [-\tau, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $w(t) \in \mathbb{R}^l$ is the disturbance input, which belongs to $L_1^n[0, +\infty)$; $z(t) \in \mathbb{R}^q$ is the controlled out; $\tau(t)$ denotes time-varying function and satisfies $0 < \tau(t) \le \tau$, $\dot{\tau}(t) \le h$, where τ, and h are known real constant scalars; $\varphi(\theta)$ is a vector-valued initial continuous function which is defined on interval $[-\tau, 0]$; $\{g_t, t \ge 0\}$ is a time-homogeneous stochastic Markovian process with right continuous trajectories, and takes values in a finite set $S = \{1, 2, \ldots, N\}$ with transition rate matrix $\Pi = \{\pi_{ij}\}$, $i, j \in S$. The transition rate from mode i at time t to mode j at time $t + \Delta t$ is given by:

$$P\{g_{t+\Delta t} = j | g_t = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & i \ne j, \\ 1 + \pi_{ij}\Delta t + o(\Delta t), & i = j, \end{cases}$$

where $\Delta t \ge 0$, $\lim\limits_{\Delta t \to 0}(o(\Delta t)/\Delta t) = 0$ and $\pi_{ij} \ge 0$, for $i \ne j$ and $\sum_{j=1, i\ne j}^{N} \pi_{ij} = -\pi_{ii}$.

Throughout the paper, the transition rates are built to be partly known, that means there are only some elements to be obtained in matrix $\Pi = \{\pi_{ij}\}$. For $\forall i \in S$, the set S^i represents $S^i = S_k^i \bigcup S_{uk}^i$, with

$$S_k^i \triangleq \{j : \pi_{ij} \text{ is known, for } j \in S\},\ S_{uk}^i \triangleq \{j : \pi_{ij} \text{ is unknown, for } j \in S\}.$$

And if $S^i \ne \varnothing$, it is further given by

$$S_k^i \triangleq \left\{k_1^i, k_2^i, \ldots, k_m^i\right\}, 1 \le m \le N,$$

where $k_m^i \in S$ means the mth known transition rate of S_k^i in the ith row of the matrix Π. For simplicity, for $g_t = i$, $A(g_t)$, $A_d(g_t)$, $G(g_t)$, $C(g_t)$ and $D(g_t)$ are, respectively, denoted as A_i, A_{di}, G_i, C_i and D_i.

The following definitions and lemmas will be introduced in the rest of this section.

Definition 1 [6] System (1) is said to be positive if, for any initial condition $\varphi(\theta) \succeq 0$, $\theta \in [-\tau, 0]$, the corresponding trajectories $x(t) \succeq 0$ and $z(t) \succeq 0$ hold for all $t > 0$.

Lemma 1 [6] *System (1) is said to be positive if and only if A_i, for all $i \in S$, are Metzler matrices and $A_{di} \succeq 0$, $G_i \succeq 0$, $C_i \succeq 0$, $D_i \succeq 0$.*

Lemma 2 [23] *A matrix is a Metzler matrix if and only if there exists a positive constant ε such that $A + \varepsilon I \succeq 0$.*

Definition 2 [23] The positive Markovian jump system (1) ($w(t) = 0$) is said to be stochastically stable if for any initial condition $\varphi(\theta)$ and $g_0 \in S$, the following inequality holds

$$E\left\{\int_0^\infty ||x(t)||_1 \mathrm{d}t|\varphi(\theta), g_0\right\} < \infty. \tag{2}$$

Definition 3 [30] For given positive scalar $\gamma > 0$, the positive Markovian jump system (1) is said to be stochastic stable with L_1-gain performance, if the following conditions are satisfied

(i) System (1) is stochastically stable when $w(t) = 0$;
(ii) Under zero-initial condition, system (1) satisfies

$$E\left\{\int_0^\infty ||z(t)||_1 \mathrm{d}t\right\} \le \gamma E\left\{\int_0^\infty ||w(t)||_1 \mathrm{d}t\right\}, \tag{3}$$

when $w(t) \ne 0$.

Definition 4 [13] Considering $V(x(t), i)$ as the Lyapunov function for the system (1), we define the weak infinitesimal operator as follows:

$$\begin{aligned}\Gamma V(x(t), i) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[&E\{V((x + \Delta t), g(t + \Delta t))|x(t), g(t) = i)\} \\ &- V(x(t), g(t) = i)].\end{aligned}$$

In this paper, the state feedback controller is designed as follows:

$$u(t) = K_i x(t), \tag{4}$$

where K_i are the controller gain matrices. Then, the control synthesis problem will be investigated for the following positive Markovian jump system

$$\begin{aligned}\dot{x}(t) &= A_i x(t) + B_i u(t) + A_{di} x(t - \tau(t)) + G_i w(t), \\ z(t) &= C_i x(t) + D_i w(t), \\ x(t + \theta) &= \varphi(\theta), \forall \theta \in [-\tau, 0].\end{aligned} \tag{5}$$

Furthermore, with the state feedback controller (4), the closed-loop Markovian jump system is denoted as:

$$\begin{aligned}\dot{x}(t) &= (A_i + B_i K_i)x(t) + A_{di} x(t - \tau(t)) + G_i w(t), \\ z(t) &= C_i x(t) + D_i w(t), \\ x(t + \theta) &= \varphi(\theta), \forall \theta \in [-\tau, 0].\end{aligned} \tag{6}$$

In this paper, our aim is to design the state feedback controller (4) under which the closed-loop Markovian jump system (6) is positive and stochastically stable with L_1-gain performance.

Remark 1 From Lemma 1, the closed-loop Markovian jump system (6) is positive if and only if $A_i + B_i K_i, \forall i \in S$ are Metzler matrices and $A_{di} \succeq 0, G_i \succeq 0, C_i \succeq 0, D_i \succeq 0$. Therefore, it is not necessary to require the unforced system (5) ($u = 0$) to be positive when considering the controller design.

3 Main Results

This section will focus on the problem of stability analysis, L_1-gain performance analysis, and state feedback controller analysis.

3.1 Stochastic Stability Analysis

Firstly, let us consider the stochastic stability analysis for system (1) $(w(t) = 0)$.

Theorem 1 *The positive Markovian jump system (1)* $(w(t) = 0)$ *with time-varying delays and partly known transition rates is stochastically stable, if there exists a set of vectors* $\nu_i, \sigma_{1i}, \sigma_{2i}, \sigma_1, \sigma_2 \in \mathbb{R}_+^n$, $\rho_{1i}, \rho_{2i}, \rho_{3i} \in \mathbb{R}^n$, *for* $i \in S$, *such that*

$$A_i^T \nu_i + \sigma_{1i} + \sigma_{2i} + \tau\sigma_1 + \tau\sigma_2 + \sum_{j \in S_k^i} \pi_{ij}(\nu_j - \rho_{1i}) \prec 0, \tag{7}$$

$$A_{di}^T \nu_i - (1-h)\sigma_{1i} \prec 0, \sum_{j \in S_k^i} \pi_{ij}(\sigma_{1j} - \rho_{2i}) - \sigma_1 \prec 0, \tag{8}$$

$$\sum_{j \in S_k^i} \pi_{ij}(\sigma_{2j} - \rho_{3i}) - \sigma_2 \prec 0, \tag{9}$$

$$\nu_j - \rho_{1i} \preceq 0, \sigma_{1j} - \rho_{2i} \preceq 0, \sigma_{2j} - \rho_{3i} \preceq 0, \quad j \in S_{uk}^i, j \neq i, \tag{10}$$

$$\nu_j - \rho_{1i} \succeq 0, \sigma_{1j} - \rho_{2i} \succeq 0, \sigma_{2j} - \rho_{3i} \succeq 0, \quad j \in S_{uk}^i, j = i. \tag{11}$$

Proof For the positive Markovian jump system (1) $(w(t) = 0)$, choose the co-positive type Lyapunov function candidate as

$$V(x(t), i) = V_1(x(t), i) + V_2(x(t), i) + V_3(x(t), i), \tag{12}$$

where

$$\begin{aligned} V_1(x(t), i) &= x^T(t)\nu_i, \\ V_2(x(t), i) &= \int_{t-\tau(t)}^{t} x^T(s)\sigma_{1i}\mathrm{d}s + \int_{t-\tau}^{t} x^T(s)\sigma_{2i}\mathrm{d}s, \\ V_3(x(t), i) &= \int_{-\tau}^{0}\int_{t+\theta}^{t} x^T(s)(\sigma_1 + \sigma_2)\mathrm{d}s\mathrm{d}\theta, \end{aligned}$$

where $\nu_i, \sigma_{1i}, \sigma_{2i}, \sigma_1, \sigma_2 \in \mathbb{R}_+^n$ and

$$\sum_{j=1}^{N} \pi_{ij}\sigma_{1j} \preceq \sigma_1, \sum_{j=1}^{N} \pi_{ij}\sigma_{2j} \preceq \sigma_2.$$

According to Definition 4, it can be shown that

$$
\begin{aligned}
\Gamma V_1(x(t),i) =& x^T(t)(A_i^T\nu_i + \sum_{j=1}^N \pi_{ij}\nu_j) + x^T(t-\tau(t))A_{di}^T\nu_i, \\
\Gamma V_2(x(t),i) =& x^T(t)\sigma_{1i} - (1-\dot{\tau}(t))x^T(t-\tau(t))\sigma_{1i} + \int_{t-\tau(t)}^t x^T(s)\sum_{j=1}^N \pi_{ij}\sigma_{1j}\mathrm{d}s \\
&+ x^T(t)\sigma_{2i} - x^T(t-\tau)\sigma_{2i} + \int_{t-\tau}^t x^T(s)\sum_{j=1}^N \pi_{ij}\sigma_{2j}\mathrm{d}s \\
\le& x^T(t)\sigma_{1i} - (1-h)x^T(t-\tau(t))\sigma_{1i} + \int_{t-\tau(t)}^t x^T(s)\sigma_1\mathrm{d}s \\
&+ x^T(t)\sigma_{2i} - x^T(t-\tau)\sigma_{2i} + \int_{t-\tau}^t x^T(s)\sigma_2\mathrm{d}s, \\
\Gamma V_3(x(t),i) =& \tau x^T(t)(\sigma_1+\sigma_2) - \int_{t-\tau}^t x^T(s)(\sigma_1+\sigma_2)\mathrm{d}s \\
\le& \tau x^T(t)(\sigma_1+\sigma_2) - \int_{t-\tau(t)}^t x^T(s)\sigma_1\mathrm{d}s - \int_{t-\tau}^t x^T(s)\sigma_2\mathrm{d}s. \qquad (13)
\end{aligned}
$$

Based on $\sum_{j=1}^N \pi_{ij}\rho_{1i} = \sum_{j=1}^N \pi_{ij}\rho_{2i} = \sum_{j=1}^N \pi_{ij}\rho_{3i} = 0$ for a set of vectors $\rho_{1i}, \rho_{2i}, \rho_{3i}$, we have

$$
\begin{aligned}
\Gamma V(x(t),i) \le& x^T(t)\left(A_i^T\nu_i + \sigma_{1i} + \sigma_{2i} + \tau\sigma_1 + \tau\sigma_2 + \sum_{j=1}^N \pi_{ij}\nu_j - \sum_{j=1}^N \pi_{ij}\rho_{1i}\right) \\
&+ x^T(t-\tau(t))\left(A_{di}^T\nu_i - (1-h)\sigma_{1i}\right) - x^T(t-\tau)\sigma_{2i} \\
=& x^T(t)\left(A_i^T\nu_i + \sigma_{1i} + \sigma_{2i} + \tau\sigma_1 + \tau\sigma_2 + \sum_{j\in S_k^i} \pi_{ij}(\nu_j - \rho_{1i}\right) \\
&+ \sum_{j\in S_{uk}^i} \pi_{ij}(\nu_j - \rho_{1i})) + x^T(t-\tau(t))\left(A_{di}^T\nu_i - (1-h)\sigma_{1i}\right) \\
&- x^T(t-\tau)\sigma_{2i}, \qquad (14)
\end{aligned}
$$

and

$$
\sum_{j=1}^N \pi_{ij}\sigma_{1j} - \sum_{j=1}^N \pi_{ij}\rho_{2i} - \sigma_1 = \sum_{j\in S_k^i} \pi_{ij}(\sigma_{1j} - \rho_{2i}) + \sum_{j\in S_{uk}^i} \pi_{ij}(\sigma_{1j} - \rho_{2i})
$$

$$
\begin{aligned}
&-\sigma_1 \preceq 0,\\
\sum_{j=1}^{N}\pi_{ij}\sigma_{2j}-\sum_{j=1}^{N}\pi_{ij}\rho_{3i}-\sigma_2 = &\sum_{j\in S_k^i}\pi_{ij}(\sigma_{2j}-\rho_{3i})+\sum_{j\in S_{uk}^i}\pi_{ij}(\sigma_{2j}-\rho_{3i})\\
&-\sigma_2 \preceq 0.
\end{aligned} \tag{15}
$$

Note that $\pi_{ii} < 0(\forall i, j \in S, i = j)$ and $\pi_{ij} \geq 0(\forall i, j \in S, i \neq j)$, therefore, if $i \in S_k^i$, inequalities (7)–(10) imply the following inequality

$$
\begin{aligned}
\Gamma V(x(t), i) \leq & x^T(t)\big(A_i^T \nu_i + \sigma_{1i} + \sigma_{2i} + \tau\sigma_1 + \tau\sigma_2 + \sum_{j\in S_k^i}\pi_{ij}(\nu_j - \rho_{1i})\big)\\
= & x^T(t)\mu_i \leq -\mu_0\|x(t)\|_{L_1} < 0,
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
\mu_0 &= \min_{i=1,2,\ldots,N}\{\min_{s=1,2,\ldots,n}\{-[\mu_i]_s\}\},\\
\mu_i &= A_i^T\nu_i + \sigma_{1i} + \sigma_{2i} + \tau\sigma_1 + \tau\sigma_2 + \sum_{j\in S_k^i}\pi_{ij}(\nu_j - \rho_{1i}),
\end{aligned}
$$

holds.

On the other hand, if $i \in S_{uk}^i$, inequalities (7)–(11) also imply that inequality (16) holds.

The use of Dynkin's formula yields

$$
\begin{aligned}
E[V(x(t), i)] - V(\varphi(\theta), g_0) = & E\left[\int_0^t \Gamma V(x(s), g(s))\mathrm{d}s\right]\\
\leq & -\mu_0 E\left[\int_0^t \|x(s)\|_1 \mathrm{d}s|(\varphi(\theta), g_0)\right],
\end{aligned}
$$

which implies

$$
\mu_0 E\left[\int_0^t \|x(s)\|_1 \mathrm{d}s|(\varphi(\theta), g_0)\right] \leq V(\varphi(\theta), g_0) - E[V(x(t), i)] \leq V(\varphi(\theta), g_0).
$$

Therefore,

$$
E\left[\int_0^\infty \|x(t)\|_1 \mathrm{d}t|(\varphi(\theta), g_0)\right] \leq \frac{V(\varphi(\theta), g_0)}{\mu_0} < \infty.
$$

The proof is completed. □

Remark 2 The free-connection weighting matrix methodology is firstly proposed to study the stability of continuous-time Markovian jump system by considering the relationship among the transition rates of the subsystems in [24], which overcomes the conservativeness of using the fixed connection weighting matrices. Here, we extend this method to free-connection weighting vectors $\rho_{1i}, \rho_{2i}, \rho_{3i}$ that could reduce some conservativeness. If the fixed connection weighting vectors are applied, sufficient conditions in Theorem 1 may lead to no solution.

Remark 3 Generally speaking, for Markovian jump system with time-varying delays, Lyapunov function is frequently chosen as follows:

$$V(x(t), i) = x^T(t)\nu_i + \int_{t-\tau(t)}^{t} x^T(s)\sigma_1 \mathrm{d}s + \int_{-\tau}^{0}\int_{t+\theta}^{t} x^T(s)\sigma_2 \mathrm{d}s\mathrm{d}\theta. \tag{17}$$

The parameter in integral term of equality (17) is mode-independent, which may lead to some conservativeness. Here, an appropriate mode-dependent co-positive type Lyapunov function (12) is constructed, and parameter in integral term $V_2(x(t), i)$ is mode-dependent, which may reduce some conservativeness.

3.2 L_1-Gain Performance Analysis

In this subsection, based on Theorem 1, some sufficient conditions will be provided to ensure stochastic stability of the system (1) with L_1-gain performance.

Theorem 2 *For given positive constant γ, if there exist a set of vectors $\nu_i, \sigma_{1i}, \sigma_{2i}, \sigma_1, \sigma_2 \in \mathbb{R}^n_+$, $\rho_{1i}, \rho_{2i}, \rho_{3i} \in \mathbb{R}^n$, for $i \in S$, such that inequalities (8)–(11) and the following inequalities*

$$A_i^T\nu_i + \sigma_{1i} + \sigma_{2i} + \tau\sigma_1 + \tau\sigma_2 + \sum_{j\in S_k^i}\pi_{ij}(\nu_j - \rho_{1i}) + C_i^T\mathbf{1} \prec 0, \tag{18}$$

$$G_i^T\nu_i + D_i^T\mathbf{1} - \gamma\mathbf{1} \prec 0. \tag{19}$$

hold, the positive Markovian jump system (1) with time-varying delays and partly known transition rates is stochastically stable with L_1-gain performance.

Proof It is clear that inequality (7) holds if inequality (18) is satisfied. Based on Theorem 1, system (1) ($w(t) = 0$) is stochastically stable.

For the closed-loop system (1) with Lyapunov function (12), we have

$$\begin{aligned}
&\Gamma V(x, i) + \|z(t)\|_{L_1} - \gamma\|w(t)\|_{L_1} \\
&\quad \le x^T(t)(A_i^T\nu_i + \sigma_{1i} + \sigma_{2i} + \tau\sigma_1 + \tau\sigma_2 + \sum_{j=1}^{N}\pi_{ij}\nu_j - \sum_{j=1}^{N}\pi_{ij}\rho_{1i}) \\
&\qquad + x^T(t-\tau(t))(A_{di}^T\nu_i - (1-h)\sigma_{1i}) - x^T(t-\tau)\sigma_{2i} + w^T(t)G_i^T\nu_i + \mathbf{1}^T C_i x(t) \\
&\qquad + \mathbf{1}^T D_i w(t) - \gamma\mathbf{1}^T w(t)
\end{aligned}$$

$$= x^T(t)(A_i^T\nu_i+\sigma_{1i}+\sigma_{2i}+\tau\sigma_1+\tau\sigma_2+\sum_{j\in S_k^i}\pi_{ij}(\nu_j-\rho_{1i})+\sum_{j\in S_{uk}^i}\pi_{ij}(\nu_j-\rho_{1i}))$$
$$+ x^T(t-\tau(t))(A_{di}^T\nu_i-(1-h)\sigma_{1i}+C_i^T\mathbf{1})-x^T(t-\tau)\sigma_{2i}$$
$$+ w^T(t)(G_i^T\nu_i+D_i^T\mathbf{1}-\gamma\mathbf{1}). \tag{20}$$

Under zero-initial condition, inequalities (8)–(11) and (18)–(19) imply that

$$\Gamma V(x,i)+\|z(t)\|_{L_1}-\gamma\|w(t)\|_{L_1}<0, \tag{21}$$

which means

$$E\left\{\int_0^\infty ||z(t)||_1\mathrm{d}t\right\}\le\gamma E\left\{\int_0^\infty ||w(t)||_1\mathrm{d}t\right\}.$$

This completes the proof of Theorem 2. □

3.3 State Feedback Controller Design

In this subsection, we will design the state feedback controller to make the corresponding closed-loop system (6) positive and stochastically stable with L_1-gain performance.

Theorem 3 *For given positive constant γ, if there exist a set of vectors $\nu_i, \sigma_{1i}, \sigma_{2i}, \sigma_1, \sigma_2 \in \mathbb{R}_+^n$, $\rho_{1i}, \rho_{2i}, \rho_{3i}, \kappa_i \in \mathbb{R}^n$ and positive constants ε_i for $i\in S$, such that inequalities* (8)–(11), (19) *and the following inequalities*

$$A_i^T\nu_i+\kappa_i+\sigma_{1i}+\sigma_{2i}+\tau\sigma_1+\tau\sigma_2+\sum_{j\in S_k^i}\pi_{ij}(\nu_j-\rho_{1i})+C_i^T\mathbf{1}\prec 0, \tag{22}$$
$$\tilde{\nu}_i^TB_i^T\nu_iA_i+B_i\tilde{\nu}_i\kappa_i^T+\varepsilon_iI\succeq 0, \tag{23}$$

hold, where $B_i\tilde{\nu}_i\succ 0$, $\tilde{\nu}_i\in\mathbb{R}_+^n$ are a set of given vectors, the closed-loop Markovian jump system (6) with time-varying delays and partly known transition rates is positive, stochastically stable with L_1-gain performance. Moreover, the state feedback controller can be given as

$$u(t)=K_ix(t)=\frac{1}{\tilde{\nu}_i^TB_i^T\nu_i}\tilde{\nu}_i\kappa_i^Tx(t). \tag{24}$$

Proof Firstly, we prove the positivity of system (6). As $B_i\tilde{\nu}_i\succ 0$ and $\nu_i\succ 0$, it is clear that $\tilde{\nu}_i^TB_i^T\nu_i\succ 0$. Dividing two sides of (23) by $\tilde{\nu}_i^TB_i^T\nu_i$ yields

$$A_i+\frac{1}{\tilde{\nu}_i^TB_i^T\nu_i}B_i\tilde{\nu}_i\kappa_i^T+\frac{1}{\tilde{\nu}_i^TB_i^T\nu_i}\varepsilon_iI\succeq 0, \tag{25}$$

From equality (24) and inequality (25), we have

$$A_i + B_i K_i + \frac{1}{\tilde{\nu}_i^T B_i^T \nu_i} \varepsilon_i I \succeq 0, \tag{26}$$

By Lemma 2, it means that $A_i + B_i K_i$ are Metzler matrices for $\forall i \in S$. Therefore, according to Lemma 1, system (6) is positive.

Then, letting $\kappa_i = K_i^T B_i^T \nu_i$ and replacing A_i with $A_i + B_i K_i$ in the inequalities (18), we can drive that the inequalities (22) hold. The rest of the proof is similar to that in Theorem 2, and we omit it here.

The proof is completed. □

Remark 4 In inequality (23), we cite the construction of the state feedback controller gain matrices K_i in the literature [23]. Then, the positivity of system can be derived through this method. The vectors $\tilde{\nu}_i$ can be given for the convenience of computation, which are chosen freely.

Remark 5 Let $\mathbf{vec}(X)$ $X \in \mathbb{R}^{m\times n}$ denote the vector formed by stacking the columns of X into one long vector: $\mathbf{vec}(X) := [x_{11}\ x_{21}\ \cdots\ x_{m1}\ x_{12}\ x_{22}\ \cdots\ x_{m2}\ x_{1n}\ x_{2n}\ \cdots\ x_{mn}]^T$. The left part of the inequality (23) is given in the form of linear matrix inequality, which can be converted into one long vector by **vec** vector operation. Then, it follows

$$\mathbf{vec}\{\tilde{\nu}_i^T B_i^T \nu_i A_i + B_i \tilde{\nu}_i \kappa_i^T + \varepsilon_i I\} \succeq 0. \tag{27}$$

Therefore, inequality (27) can be computed in linear programming.

For system (6) in the single-input form, we directly give the following corollary. Its proof can be obtained by using a method similar to that in Theorem 3.

Corollary 1 *For given positive constant γ, if there exist a set of vectors ν_i, σ_{1i}, σ_{2i}, σ_1, $\sigma_2 \in \mathbb{R}_+^n$, ρ_{1i}, ρ_{2i}, ρ_{3i}, $\kappa_i \in \mathbb{R}^n$ and positive constants ε_i for $i \in S$, such that inequalities (8)–(11), (19), (22) and the following inequality*

$$B_i^T \nu_i A_i + B_i \kappa_i^T + \varepsilon_i I \succeq 0, \tag{28}$$

hold, the closed-loop Markovian jump system (6) with time-varying delays and partly known transition rates is positive, stochastically stable with L_1-gain performance. Moreover, the state feedback controller can be given as

$$u(t) = K_i x(t) = \frac{1}{B_i^T \nu_i} \kappa_i^T x(t). \tag{29}$$

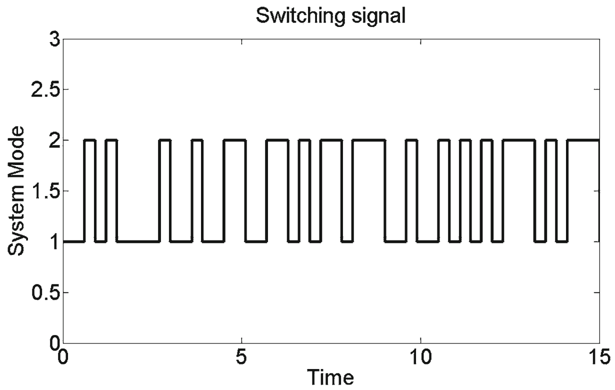

Fig. 1 System mode

4 Numerical Example

Consider two-mode Markovian jump system with parameters as follows:

$$A_1=\begin{bmatrix}-1 & 0.2\\ 0.1 & 0.2\end{bmatrix}, A_2=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}, A_{d1}=\begin{bmatrix}0.2 & 0.2\\ 0.1 & 0.2\end{bmatrix}, A_{d2}=\begin{bmatrix}0.1 & 0.1\\ 0.1 & 0.2\end{bmatrix}, B_1=\begin{bmatrix}0.1 & 0.1\\ 0.3 & 0.5\end{bmatrix},$$
$$B_2=\begin{bmatrix}0.2 & 0.2\\ 0.2 & 0.3\end{bmatrix}, G_1=\begin{bmatrix}0.3 & 0.2\\ 0.4 & 0.2\end{bmatrix}, G_2=\begin{bmatrix}0.1 & 0.4\\ 0.3 & 0.1\end{bmatrix}, C_1=\begin{bmatrix}0.3 & 0.3\\ 0.1 & 0.2\end{bmatrix}, C_2=\begin{bmatrix}0.2 & 0.1\\ 0.1 & 0.2\end{bmatrix},$$
$$D_1=\begin{bmatrix}0.2 & 0.2\\ 0.3 & 0.2\end{bmatrix}, D_2=\begin{bmatrix}0.4 & 0.2\\ 0.4 & 0.3\end{bmatrix}, \gamma=1, \tilde{\nu}_1=\begin{bmatrix}-1\\ 1\end{bmatrix}, \tilde{\nu}_2=\begin{bmatrix}-1\\ 1\end{bmatrix}.$$

Consider the partly known transition rate matrix as follows:

$$\begin{bmatrix}? & ?\\ 0.3 & -0.3\end{bmatrix}.$$

Let $\tau(t)=0.5(1-sin(t))$, then $\tau=0.5$, $\dot{\tau}(t)=-0.5cos(t)$, $h=0.5$.

Choosing the disturbance input $w(t)=\begin{bmatrix}e^{-t}(2-sin(t)) & e^{-t}(2-sin(t))\end{bmatrix}^{\mathrm{T}}$, simulation results are shown in Figs. 1, 2 and 3 for initial conditions $g(0)=1$ and $x(0)=\begin{bmatrix}0.5 & 1\end{bmatrix}^{\mathrm{T}}$, $x(t)=\begin{bmatrix}0 & 0\end{bmatrix}^{\mathrm{T}}$, $t\in[-\tau,0)$.

Figures 1, 2, and 3 stand for the system mode, state trajectory and controlled output of the open-loop systems with time-varying delays and partly known transition rates. It is easily seen that the open-loop positive Markovian jump system with time-varying delays and partly known transition rates is unstable.

Solving Theorem 3, we get the controller gain matrices as follows:

$$K_1=\begin{bmatrix}-0.5218 & 27.5592\\ 0.5218 & -27.5592\end{bmatrix}, K_2=\begin{bmatrix}-0.4649 & 158.3962\\ 0.4649 & -158.3962\end{bmatrix}.$$

Figures 4 and 5 show the state trajectory $x(t)$ and the controlled output $z(t)$ of the closed-loop system. It is shown that the closed-loop Markovian jump system with

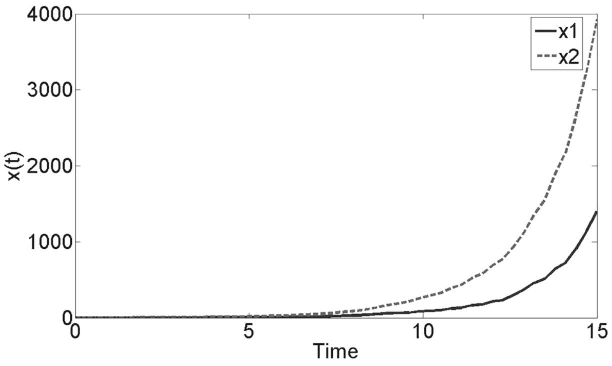

Fig. 2 State trajectory of the open-loop system with partly known transition rates

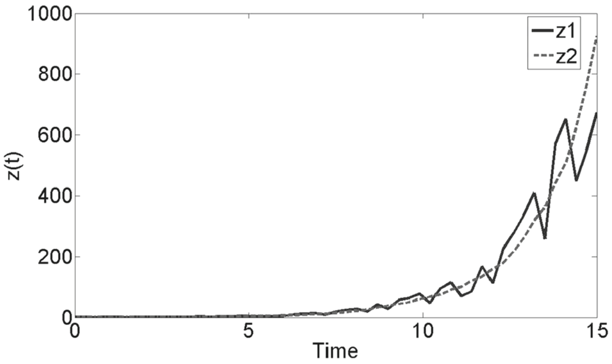

Fig. 3 Controlled output of the open-loop system with partly known transition rates

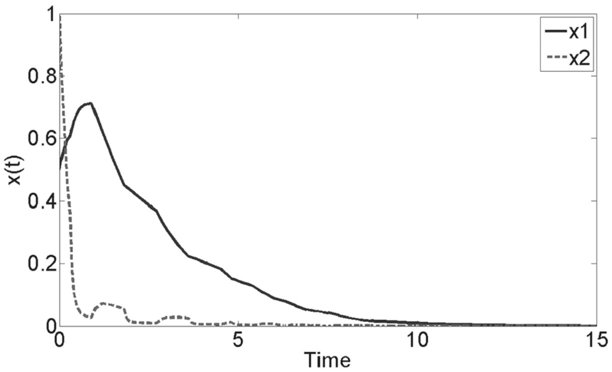

Fig. 4 State trajectory of the closed-loop system with partly known transition rates

time-varying delays and partly known transition rates is positive and stochastically stable with L_1-gain performance.

Remark 6 If Lyapunov function is chosen in the form of equality (17), we cannot get the state feedback gain matrices. This means that the chosen Lyapunov function (12) reduces some conservativeness.

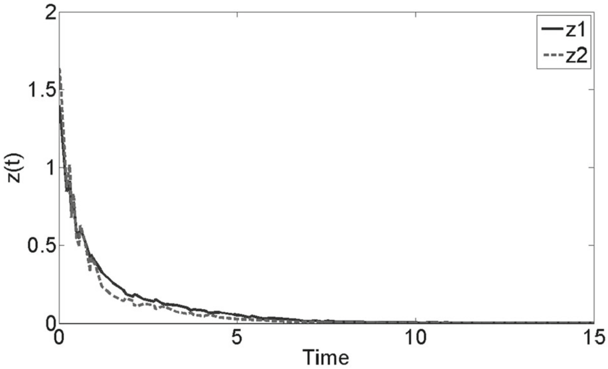

Fig. 5 Controlled output of the closed-loop system with partly known transition rates

Remark 7 If the complete known transition rate matrix is given as follows:

$$\begin{bmatrix} -0.4 & 0.4 \\ 0.3 & -0.3 \end{bmatrix},$$

then we get the controller gain matrices by solving Theorem 3 as follows:

$$K_1 = \begin{bmatrix} -0.8153 & 36.2644 \\ 0.8153 & -36.2644 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.5945 & 106.6321 \\ 0.5945 & -106.6321 \end{bmatrix}.$$

It means that the closed-loop Markovian jump system with time-varying delays and complete known transition rates is positive and stochastically stable with L_1-gain performance.

5 Conclusions

In this paper, we have given an approach for the design of L_1-gain state feedback controller for Markovian jump systems with time-varying delays and partly known transition rates. By using appropriate co-positive type Lyapunov function, sufficient conditions, which ensure the closed-loop system is positive and stochastically stable with L_1-gain performance, are given in linear programming. For positive Markovian jump systems with time-varying delays, there are still many challenging problems to be solved such as mode-dependent time-varying delays, filter design, reliable control, and other aspects. It is expected that the ideas in this paper will be helpful for future work in the corresponding fields.

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