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Brief paper

Finite-time control of robotic manipulators ☆

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ABSTRACT

This work offers the solution at the control feed-back level of the accurate trajectory tracking subject to finite-time convergence. Dynamic equations of a rigid robotic manipulator are assumed to be uncertain. Moreover, globally unbounded disturbances are allowed to act on the manipulator when tracking the trajectory. Based on the suitably defined non-singular terminal sliding vector variable and the Lyapunov stability theory, we propose a class of absolutely continuous robust controllers which seem to be effective in counteracting both uncertain dynamics and unbounded disturbances. The numerical simulation results carried out for a robotic manipulator consisting of two revolute kinematic pairs operating in a two-dimensional joint space illustrate performance of the proposed controllers.

1. Introduction

Present-day robotic tasks require high precision and stability of their performance. Trajectory tracking seems to be a fundamental task in robot control. In order to fulfil aforementioned requirements, control algorithms should take into account the following factors: model uncertainties, parameter variations and external disturbances. However, they are, in fact, never known exactly in practice. Therefore, it is particularly important to design control algorithms that ensure accurate and fast convergence to the stable equilibrium when trajectory tracking despite the existence of the aforementioned factors. In such a context, several control schemes for asymptotic tracking of manipulator trajectories can be found in the literature (Corless, 1993; Galicki, 2008, 2012; Hsu & Fu, 2006; Slotine & Li, 1991; Utkin, 1978; Zhang, Dawson, de Queiroz, & Dixon, 2000) which partially or fully take into account these factors. Sliding-mode control seems to be one of the most effective approaches to cope with uncertainties. As is well known, sliding mode is accurate and insensitive to disturbances (Edwards & Spurgeon, 1998; Utkin, 1992). However, the main drawback of the standard first-order sliding modes is mostly related to the undesirable chattering effect (Fridman, 2002). The second- and higher-order sliding techniques to eliminate the chattering have been proposed (Bartolini, Ferrara, & Punta, 2000; Bartolini, Ferrara, Usai, & Utkin, 2000; Bartolini, Pisano, Punta, & Usai, 2003; Bartolini & Pydynowski, 1996; Ferrara & Capisani, 2011; Levant, 1998, 2003, 2005, 2011; Levant & Michael, 2009; Mondal & Mahanta, 2014; Shtessel, Shkolnikov, & Brown, 2003; Sira-Ramírez, 1992). Nevertheless, the approaches from Bartolini, Ferrara, Punta (2000), Bartolini, Ferrara, Usai et al. (2000), Bartolini et al. (2003), Bartolini and Pydynowski (1996), Ferrara and Capisani (2011), Mondal and Mahanta (2014), Shtessel et al. (2003) and Sira-Ramírez (1992) are able to steer a tracking error to zero asymptotically and those from Levant (1998, 2003, 2005, 2011) and Levant and Michael (2009) are only applicable to single input dynamic systems. In order to both increase tracking accuracy and accelerate a convergence process to the stable equilibrium, terminal sliding mode (TSM) control techniques have been offered as a particularly useful tool for high precision control of robotic manipulators. In such a context, several approaches can be distinguished (Hong, Xu, & Huang, 2002; Su, 2009; Su & Zheng, 2011) that produce (non-smooth) continuous controls but require the full knowledge of robot dynamic equations. By using the regressor matrix technique, adaptive-discontinuous TSM controllers have been designed in works Barambones and Etxebarria (2002), Parra-Vega, Rodrigues-Angeles, and Hirzinger (2001) and Tang (1998). An alternative terminal sliding manifold has been proposed in Feng, Yu, and Man (2002), Jin, Lee, Chang, and Choi (2009) and Yu, Yu, Shirinzadeh, and Man (2005) to eliminate the singularity problem. Nevertheless, the common feature of the approaches from Feng et al. (2002), Jin et al. (2009) and Yu et al. (2005) is necessity of knowledge of the nominal robot dynamic equations whose construction may not be a trivial task. Recently, a robust discontinuous TSM control for robotic manipulators has been proposed in Zhao, Li, and Gao (2009). A similar approach with a singularity problem has also been presented in Man, Paplinski, and Wu (1994). From the

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literature survey, it follows that all the aforementioned algorithms are not able to generate continuous controls resulting in finite-time stability of the equilibrium when both dynamic equations are uncertain and (unbounded) disturbances act on the robotic manipulators. Hence, there is a need to provide additional information (including the joint position and velocity or its estimation) for a control scheme to be designed further on. From the robotic point of view, joint acceleration is such additional quantity. In general, there are two approaches for the joint acceleration acquisition. The first is based on the direct measurement of joint acceleration (De Luca, Schroder, & Thummel, 2007; Godler, Akahane, Maruyama, & Yamashita, 1995). The second approach uses a class of uniform robust differentiators (Levant, 2003; Levant & Livne, 2012). Based on the available joint acceleration or its estimation, a new non-singular TSM manifold is introduced in this study. The proposed TSM manifold makes it possible to simultaneously join the first-order sliding mode approach possessing the finite-time control capabilities with the second-order sliding mode techniques generating the (absolutely) continuous controls. It is worth to emphasise that the finite-time control of robotic manipulators subject to uncertain dynamic equations, absolute continuity control requirement and globally unbounded disturbances, is still a non-trivial problem whose solution is based in this work on introducing a dynamic version of a static computed torque approach presented in e.g. works Siciliano, Sciavicco, Villani, and Oriolo (2009) and Spong and Vidyasagar (1989). The remainder of the paper is organised as follows. Section 2 formulates the finite-time trajectory tracking task. Section 3 sets up a class of robust absolutely continuous controllers solving the trajectory tracking task in a finite-time subject to uncertain robot dynamic equations and unbounded disturbances. Section 4 presents computer examples of trajectory tracking by a robotic manipulator consisting of two revolute kinematic pairs. Finally, some concluding remarks are drawn in Section 5. Throughout this paper, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimal and maximal, respectively, eigenvalues of the symmetric matrix $(\cdot)$. Moreover, the real branch of $x^{\frac{a}{b}}$, where $x \in \mathbb{R}$; a, b are positive odd numbers, and $a < b < 2a$, is taken here into account.

2. Problem formulation

The dynamics of a rigid robotic manipulator of n-DoF is given by the following general equations (Spong & Vidyasagar, 1989):

$$M(q)\ddot{q} + H(q,\ \dot{q}) + G(q) + D(t,\ q,\ \dot{q}) = v, \tag{1}$$

where $q = (q_1, \ldots, q_n)^T$, $\dot{q}$ and $\ddot{q}$ represent the position, velocity and acceleration, respectively. The $n \times n$ inertia matrix $M(q)$ is positive definite and symmetric. The term H in (1) equals $H = B(q)(\dot{q} \cdot \dot{q}) + C(q)(\dot{q}^2)$, where B and C are the $n \times \frac{n(n-1)}{2}$ and $n \times n$ matrices of coefficients of the Coriolis and centrifugal forces, respectively. Quantities $(\dot{q} \cdot \dot{q})$ and $(\dot{q}^2)$ are the symbolic notations for the $\frac{n(n-1)}{2}$-dimensional and n-dimensional vectors $(\dot{q} \cdot \dot{q}) = (\dot{q}_1\dot{q}_2, \ldots, \dot{q}_{n-1}\dot{q}_n)^T$ and $(\dot{q}^2) = (\dot{q}_1^2, \ldots, \dot{q}_n^2)^T$, respectively. Term $v = (v_1, \ldots, v_n)^T$ stands for the n-dimensional vector of controls (torques/forces). Term $G(q)$ is the n-dimensional vector of generalised gravity forces. Vector $D(t,\ q,\ \dot{q})$ means the n-dimensional external disturbance signal which is (by assumption) at least absolutely continuous mapping with $\dot{D}(t,\ q,\ \dot{q})$ as being a locally bounded Lebesgue measurable mapping (this implies existence of control v). Moreover, $\|D\|$ and $\|\dot{D}\|$ are (by assumption) upper estimated as follows

$$\|D\| \le \alpha_0(t), \qquad \|\dot{D}\| \le \alpha_1(t), \tag{2}$$

where α_0, α_1 stand for the known, non-negative functions. In the sequel, useful properties of (1) are summarised which will be utilised while designing the controller. The following inequalities are satisfied (Spong & Vidyasagar, 1989):

$$\begin{aligned} &0 < \lambda_{\min}(M^{-1}) \le \|M^{-1}\| \le \lambda_{\max}(M^{-1}), \\ &\|B + C\| \le c_1, \qquad \|G\| \le c_2, \end{aligned} \tag{3}$$

where c_1, c_2 are known positive scalar coefficients. In order to obtain at least absolutely continuous control v, let us differentiate the dynamic equations (1) with respect to time

$$M(q)\frac{d^3q}{dt^3} + F(q,\ \dot{q},\ \ddot{q},\ t) = \dot{v}, \tag{4}$$

where $F = \dot{M}\ddot{q} + \dot{B}(\dot{q}\cdot\dot{q}) + \dot{C}(\dot{q}^2) + B\frac{d}{dt}(\dot{q}\cdot\dot{q}) + C\frac{d}{dt}(\dot{q}^2) + \dot{G} + \dot{D}$. Based on the properties of (1), one obtains the following upper estimation of $\|F\|$:

$$\|F\| \le \mathcal{E}(q,\ \dot{q},\ \ddot{q},\ t), \tag{5}$$

where $\mathcal{E} = c_3\|\dot{q}\|\|\ddot{q}\| + c_4\|\dot{q}\|^3 + c_5\|\dot{q}\| + \alpha_1(t)$; c_3, c_4, c_5 are (known by assumption) positive scalar coefficients for which the following inequalities hold true: $\|\frac{\partial M}{\partial q}\| + \|B\| + \|C\| \le c_3$; $\|\frac{\partial C}{\partial q}\| \le c_4$; $\|\frac{\partial G}{\partial q}\| \le c_5$. Motivated in part by the static computed torque methodology (Siciliano et al., 2009; Spong & Vidyasagar, 1989), we propose now a dynamically computed torque vector $\dot{v}$ of the form

$$\dot{v} = \hat{M}(q)u + \hat{F}(q,\ \dot{q},\ \ddot{q},\ t), \tag{6}$$

where $\hat{M}$ and $\hat{F}$ denote known estimates of the corresponding unknown terms M and F, respectively, in dynamic equations (4); $u \in \mathbb{R}^n$ is a new control to be found. The use of (6) as a dynamic non-linear control law gives $M\frac{d^3q}{dt^3} + F = \hat{M}u + \hat{F} = \dot{v}$. Since M is invertible, we obtain

$$\frac{d^3q}{dt^3} = u + (\mathcal{R} - \mathbb{I}_n)u + Q, \tag{7}$$

where $\mathcal{R} = M^{-1}\hat{M}$; $Q = M^{-1}(\hat{F} - F)$; $\mathbb{I}_n$ stands for the $n \times n$ identity matrix. A task accomplished by the robotic manipulator consists in tracking a desired trajectory $q_d(t) \in \mathbb{R}^n, t \in [0,\ \infty)$ which is assumed to be at least triply continuously differentiable, i.e., $q_d(\cdot) \in C^3[0,\ \infty)$. By introducing the tracking error $e = (e_1, \ldots, e_n)^T = q - q_d(t)$, we may formally express the finite-time trajectory tracking control by means of the following equations:

$$\lim_{t\to T} e(t) = \lim_{t\to T} \dot{e}(t) = \lim_{t\to T} \ddot{e}(t) = 0, \tag{8}$$

where $0 \le T$ denotes a finite time of convergence of q to q_d. The objective is to find an input signal $u(t)$ and consequently a control vector $v(t)$ by solving the differential equations (6) such that position vector q follows q_d. The next section will present an approach to the solution of the control problem (6)–(8) making use of the Lyapunov stability theory.

3. Control of the robotic manipulator

In the sequel, we start the analysis of a controller design by the assumption that joint positions, velocities and accelerations are available from measurements. Let us note that the right-hand side of (7) requires the knowledge of joint acceleration $\ddot{q}$. Recently, a lot of techniques appeared in the literature which directly measure $\ddot{q}$ (De Luca et al., 2007; Godler et al., 1995). Based on (3), we can make the following remark:

$$(\exists\hat{M} > 0)(\exists\rho > 0)(\|\mathcal{R} - \mathbb{I}_n\| \le \rho < 1). \tag{9}$$

Let us note that it is not difficult to find matrix $\hat{M}$ fulfilling relations (9). If we set $\hat{M} = \frac{2}{\lambda_{\min}(M^{-1}) + \lambda_{\max}(M^{-1})}\mathbb{I}_n$ (see e.g. Spong & Vidyasagar, 1989) then $\rho = \frac{\lambda_{\max}(M^{-1}) - \lambda_{\min}(M^{-1})}{\lambda_{\max}(M^{-1}) + \lambda_{\min}(M^{-1})}$ satisfies inequality

(9). Furthermore, based on (4), (5) and definition of Q in (7), an upper estimation on $\|Q\|$ takes the form

$$\|Q\| \le \mathcal{W}(q,\ \dot{q},\ \ddot{q},\ t), \tag{10}$$

where $\mathcal{W} = \lambda_{\max}(M^{-1})(\|\hat{F}\| + \mathcal{E})$. Before we propose our controller and show its properties, some useful inequality will now be given. For arbitrary $\overline{a} = (a_1, \ldots, a_L)^T \in \mathbb{R}^L, L \ge 1$, the following relation holds true (Yu et al., 2005):

$$\|\overline{a}^{\beta}\|^2 = \sum_{i=1}^{L}(a_i^2)^{\beta} \ge \|\overline{a}\|^{2\beta} = \left(\sum_{i=1}^{L} a_i^2\right)^{\beta}, \tag{11}$$

where $\overline{a}^{\beta} = (a_1^{\beta}, \ldots, a_L^{\beta})^T$; $\beta = \frac{a}{b}$. Let $s = (s_1, \ldots, s_n)^T \in \mathbb{R}^n$ be the sliding vector variable. In order to overcome the limitations and shortcomings of the first-order classic sliding variables (Feng et al., 2002; Yu et al., 2005; Zhao et al., 2009), we propose the following non-singular terminal sliding manifold:

$$s = \ddot{e} + \int_0^t \left(\lambda_2 \ddot{e}^{3/5} + \lambda_2 \lambda_1^{3/5}(\dot{e}^{9/7} + \lambda_0^{9/7} e)^{1/3}\right) d\tau, \tag{12}$$

where $\lambda_0 = \mathrm{diag}(\lambda_{0,1}, \ldots, \lambda_{0,n})$; $\lambda_1 = \mathrm{diag}(\lambda_{1,1}, \ldots, \lambda_{1,n})$; $\lambda_2 = \mathrm{diag}(\lambda_{2,1}, \ldots, \lambda_{2,n})$; $\lambda_{i,j}$ stand for positive coefficients (controller gains); $i = 0 : 2$; $j = 1 : n$. The potency of both $e, \dot{e}, \ddot{e}$ and $\lambda_0, \lambda_1, \lambda_2$ is defined component-wise. In what follows, we give a useful result.

Lemma 1. *If $s = 0$ then task errors $(e,\ \dot{e},\ \ddot{e})$ converge in a finite time to the origin $(e,\ \dot{e},\ \ddot{e}) = (0,\ 0,\ 0)$.*

Proof. Identity $s = 0$ implies equality $\dot{s} = 0$. From (12), it follows that

$$\frac{d^3 e}{dt^3} + \lambda_2 \ddot{e}^{3/5} + \lambda_2 \lambda_1^{3/5}(\dot{e}^{9/7} + \lambda_0^{9/7} e)^{1/3} = 0. \tag{13}$$

Expression (13) presents a known homogeneous triple integrator system of negative degree equal to $-\frac{2}{9}$. The finite-time stability of homogeneous system (13) was studied e.g. in Bhat and Bernstein (2000) and Hong (2002). Moreover, the settling-time estimation and the explicit form of the Lyapunov function candidate for (13) have also been given in Hong (2002). Consequently, task errors $(e,\ \dot{e},\ \ddot{e})$ converge for $s = 0$ in a finite time to the origin $(e,\ \dot{e},\ \ddot{e}) = (0,\ 0,\ 0)$. □

In order to fulfil equality constraints (8), a (simple) robust control law is proposed as follows

$$u = u_n + u_r, \tag{14}$$

where

$$u_n = \frac{d^3 q_d}{dt^3} - \lambda_2 \ddot{e}^{3/5} - \lambda_2 \lambda_1^{3/5}(\dot{e}^{9/7} + \lambda_0^{9/7} e)^{1/3} - \Lambda s^{\beta}, \tag{15}$$

$\Lambda = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_n)$; Λ_i denote constant, positive controller gains and

$$u_r = \begin{cases} -\dfrac{\kappa}{1-\rho}\dfrac{s}{\|s\|}(\rho\|u_n\| + \mathcal{W}) & \text{for } s \ne 0 \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

κ is a positive constant gain to be specified further on. Consequently, absolutely continuous control vector v can be found by solving in the Filippov sense (Filippov, 1988), the following differential equations with u_n and u_r given by (15)–(16):

$$\dot{v} = \hat{M}(u_n + u_r) + \hat{F}(q,\ \dot{q},\ \ddot{q},\ t). \tag{17}$$

The aim is to provide conditions on controller gains $\lambda_0, \lambda_1, \lambda_2, \Lambda$ and κ, which guarantee fulfilment of equalities (8). Applying the Lyapunov stability theory, we now derive the following result.

Theorem 1. *If $q, \dot{q}, \ddot{q}$ are available and $\lambda_0, \lambda_1, \lambda_2, \Lambda > 0, \kappa > 1$ then control scheme (14)–(17) guarantees stable convergence in a finite time of the tracking errors $(e,\ \dot{e},\ \ddot{e})$ to the origin $(e,\ \dot{e},\ \ddot{e}) = (0,\ 0,\ 0)$.*

Proof. Consider the following Lyapunov function candidate:

$$V = \frac{1}{2}\langle s,\ s\rangle, \tag{18}$$

where $\langle\ ,\ \rangle$ stands for the scalar product of vectors. Differentiating (18) with respect to time and taking into account definition (12) results in the following expression: $\dot{V} = \langle s,\ \frac{d^3 e}{dt^3} + \lambda_2 \ddot{e}^{3/5} + \lambda_2 \lambda_1^{3/5}(\dot{e}^{9/7} + \lambda_0^{9/7} e)^{1/3}\rangle$. Based on (7), (14) and definition of task error e, one obtains that

$$\frac{d^3 e}{dt^3} = u_n + u_r + (\mathcal{R} - \mathbb{I}_n)(u_n + u_r) + Q - \frac{d^3 q_d}{dt^3}. \tag{19}$$

Inserting the right-hand side of (19) into $\dot{V}$ results in

$$\begin{aligned} \dot{V} = &\left\langle s, u_n - \frac{d^3 q_d}{dt^3} + \lambda_2 \ddot{e}^{3/5} + \lambda_2 \lambda_1^{3/5}(\dot{e}^{9/7} + \lambda_0^{9/7} e)^{1/3}\right\rangle \\ &+ \langle s, u_r\rangle + \langle s, (\mathcal{R} - \mathbb{I}_n)(u_n + u_r) + Q\rangle. \end{aligned} \tag{20}$$

Let us estimate the sum of the last two terms of $\dot{V}$. Substituting u_r into (20) for the right-hand side of (16), we have after simple algebra that

$$\begin{aligned} &\langle s, u_r\rangle + \langle s, (\mathcal{R} - \mathbb{I}_n)(u_n + u_r) + Q\rangle \\ &\le \|s\|(\rho\|u_n\| + \mathcal{W})\left(-\frac{\kappa}{1-\rho} + 1 + \frac{\rho\kappa}{1-\rho}\right). \end{aligned} \tag{21}$$

Based on the assumption of Theorem 1 for κ, the last expression of (21) is non-positive for arbitrary both u_n and $\mathcal{W} \ge 0$. Hence,

$$\langle s, u_r\rangle + \langle s, (\mathcal{R} - \mathbb{I}_n)(u_n + u_r) + Q\rangle \le 0. \tag{22}$$

Inserting the right-hand side of (15) into (20) and taking into account inequality (22) result in the following expression:

$$\dot{V} \le -\sum_{i=1}^{n} \Lambda_i s_i^{1+\beta} \le -\min_i\{\Lambda_i\}\left\|s^{\frac{1+\beta}{2}}\right\|^2. \tag{23}$$

Applying (11) to inequality (23) results in

$$\dot{V} \le -\min_i\{\Lambda_i\}\left(\sum_{i=1}^{n} s_i^2\right)^{\frac{1+\beta}{2}} = -\min_i\{\Lambda_i\} 2^{\frac{1+\beta}{2}} V^{\frac{1+\beta}{2}}. \tag{24}$$

Since $\min_i\{\Lambda_i\} > 0$, expression (24) proves that TSM $s = 0$ is attainable in a finite time less or equal to $\frac{2V(0)^{\frac{1-\beta}{2}}}{\min_i\{\Lambda_i\}(1-\beta)2^{\frac{1+\beta}{2}}}$. Consequently, from Lemma 1, it follows that the origin $(e,\ \dot{e},\ \ddot{e}) = (0,\ 0,\ 0)$ is attainable in a finite time T. □

A few remarks may be made regarding the control law (14)–(17) and Theorem 1.

- Remark 1. Observe that controller gain of u_r given by (16) is a feed-back adjustable function equal to $\rho\|u_n\| + \mathcal{W}$. The control laws known from the literature (see e.g. Bartolini, Ferrara, Punta, 2000, Bartolini, Ferrara, Usai et al., 2000, Bartolini et al., 2003, Ferrara & Capisani, 2011, Siciliano et al., 2009 and Spong & Vidyasagar, 1989) require boundedness of $\dot{q}$ which implies large controller gains to cope with the uncertainty over the whole operation region.

- Remark 2. It is also worth to notice that our feed-back adjustable amplitude term $\frac{\kappa}{1-\rho}(\rho\|u_n\|+\mathcal{W})$ makes it possible to cope with globally unbounded uncertainties. In general, in that case, only local uncertainty suppression is available in the literature for multi-input systems. In such a context, a class of gain-function robust controllers with single input and adjustable amplitude was recently proposed in works Levant (2011) and Levant and Livne (2012) to overcome globally unbounded uncertainty problem.

A very computationally efficient approach based on the uniform robust exact differentiation has been recently proposed in works Levant (2003) and Levant and Livne (2012) to numerically find derivatives of absolutely continuous functions. Assuming that position $q = q(t)$ is known (measurable), one can exactly reconstruct both velocity $\dot q(t)$ and acceleration $\ddot q(t)$ (by neglecting the measurement noise of a device) after a finite-time transient process, say $T' > 0$. The second-order uniform robust exact differentiator takes in our case the following form:

$$\begin{aligned}
\dot y_0 &= y_1 - \hat\lambda_2 L(t)^{1/3}|y_0 - q|^{2/3}\mathrm{sign}(y_0 - q),\\
\dot y_1 &= y_2 - \hat\lambda_1 L(t)^{2/3}|y_0 - q|^{1/3}\mathrm{sign}(y_0 - q),\\
\dot y_2 &= -\hat\lambda_0 L(t)\mathrm{sign}(y_0 - q),
\end{aligned} \tag{25}$$

where $\hat\lambda_0, \hat\lambda_1, \hat\lambda_2$ are positive constants equal to $\hat\lambda_0 = 1.1, \hat\lambda_1 = 1.5$ and $\hat\lambda_2 = 2$ (as suggested by Levant & Livne, 2012), respectively; $y_0(t), y_1(t), y_2(t) \in \mathbb{R}^n$; y_1, y_2 denote the outputs of differentiator (25) reconstructing exactly both velocity $\dot q(t)$ and acceleration $\ddot q(t)$, i.e., $\dot q(t) = y_1(t), \ddot q(t) = y_2(t)$ for $t \geq T'$; $L(t)$ stands for a positive continuous function which takes the form $L(t) = L'(t) + L''(t), L' = \lambda_{\max}(M^{-1})[\lambda_{\max}(M)(\|u_n\|(1+\frac{\kappa}{1-\rho}) + \mathcal{W}) + \|\hat F\|], L'' = \lambda_{\max}(M^{-1})\{c_3\|y_1\|\lambda_{\max}(M^{-1})[\|v\| + c_1\|y_1\|^2 + c_2 + \alpha_0(t)] + c_4\|y_1\|^3 + c_5\|y_1\| + \alpha_1(t)\}$. The quantity $L(t)$ represents physically an upper estimation of the norm of $\frac{d^3q}{dt^3}$ (manipulator jerk). Let us define concatenating control $v_c = (v_{c,1}, \ldots, v_{c,n})^T$ as follows

$$v_c = \begin{cases} v'(t), & t \in [0,\ T'],\\ v(t) & \text{given by (17)},\ \ddot q(t) = y_1,\ t > T', \end{cases} \tag{26}$$

where $v'(t)$ is arbitrary absolutely continues mapping of time t (e.g. $v'(t) = 0$). Note that for $t \geq 0$, $\dot q(t)\ \ddot q(t)$ are replaced by their estimates $y_1(t), y_2(t)$ in $\mathcal{W}, \hat F, \alpha_0(t), \alpha_1(t), u_n$ and for $t \leq T'$ v is replaced by (constant) v' in $L(t)$ which implies its continuity. Based on (25) and (26), we are now in position to give the following theorem.

Theorem 2. *If q is only available from measurements and $\lambda_0, \lambda_1, \lambda_2, \Lambda > 0, \kappa > 1$ then control scheme (26) guarantees stable convergence in a finite time of the tracking errors $(e,\ \dot e,\ \ddot e)$ to the origin $(e,\ \dot e,\ \ddot e) = (0,\ 0,\ 0)$.*

Proof. Inserting v' into dynamic equations (1) results in measured joint positions $q = q(t)$ which serve as inputs to differentiator (25). For $t > T'$, one obtains $\dot q(t) = y_1(t)$ and $\ddot q(t) = y_2(t)$, respectively. Hence, control $v(t)$ defined by (14)–(17) can be applied with initial conditions $v(T') = v'(T')$ and $\dot q(T') = y_1(T'), \ddot q(T') = y_2(T')$ to track q_d. From Theorem 1, it follows that $s = 0$ is attainable in a finite time less or equal to $\frac{2V(T')^{\frac{1-\beta}{2}}}{\min_i\{\Lambda_i\}(1-\beta)2^{\frac{1+\beta}{2}}}$. Finally, from Lemma 1, it follows that the origin $(e,\ \dot e,\ \ddot e) = (0,\ 0,\ 0)$ is attained in a finite time. □

- Remark 3. In a general case, if measured position $q = q(t)$ is additionally contaminated by a measurement noise $\eta(t)$, i.e., $q(t) = q_0(t) + \eta(t)$, where $\|\eta = (\eta_1, \ldots, \eta_n)^T\| \leq \epsilon L(t)$; ϵ denotes a normalised noise magnitude (practically $\epsilon \in [10^{-5},\ 10^{-4}]$); $q_0(t)$ stands for an unknown true (noise-free) joint position; then also differentiator (25) should be applied to estimate quantities $e, \dot e$ and $\ddot e$. Notice from the differentiator equations (25) that $\|q_0(t) - y_0(t)\| \leq L(t)O(\epsilon), \|\dot q_0(t) - y_1(t)\| \leq L(t)O(\epsilon^{2/3}), \|\ddot q_0(t) - y_2(t)\| \leq L(t)O(\epsilon^{1/3})$, Levant (2003) and Levant and Livne (2012). Finally, for estimations $\hat e = (\hat e_1, \ldots, \hat e_n)^T = y_0 - q_d, \hat{\dot e} = y_1 - \dot q_d, \hat{\ddot e} = y_2 - \ddot q_d$, we obtain $\|\hat e\| \leq L(t)O(\epsilon), \|\hat{\dot e}\| \leq L(t)O(\epsilon^{2/3}), \|\hat{\ddot e}\| \leq L(t)O(\epsilon^{1/3})$.
- Remark 4. The methodology proposed in our manuscript may also be applicable to globally unbounded, continuous and everywhere non-differentiable disturbances $D'(t,\ q,\ \dot q)$ (e.g. to a Brownian motion). For this purpose, a non-singular sliding manifold $s' \in \mathbb{R}^n$ can be defined as follows $s' = \dot e + \int_0^t (\lambda_1(\dot e)^{\alpha_2} + \lambda_0 e^{\alpha_1})\, d\tau$, where α_1 is defined similarly as β and $\alpha_2 = \frac{2\alpha_1}{1+\alpha_1}$. Then, based on the simplified versions of Lemma 1 and Theorem 1, we can derive the following control law: $\tilde v = v_n + v_r$, where $v_n = \ddot q_d - \lambda_0 e^{\alpha_1} - \lambda_1(\dot e)^{\alpha_2} - \Lambda(s')^{\beta}$ and $v_r = \begin{cases} -\dfrac{\kappa}{1-\rho}\dfrac{s'}{\|s'\|}(\rho\|v_n\| + \mathcal{W}') & \text{for } s' \neq 0\\ 0 & \text{otherwise;}\end{cases}$ $\mathcal{W}' = \lambda_{\max}(M^{-1})(\|\hat F'\| + \mathcal{E}')$; $\hat F'$ denotes known estimate of the term $H + G$ of dynamic equations (1); $\mathcal{E}' = c_2\|\dot q\|^2 + c_3 + \alpha(t)$; $\|D'\| \leq \alpha(t)$ is known upper estimation of $\|D'\|$. However, the price of the assumption of the everywhere non-differentiable disturbance is discontinuity of control $\tilde v$ which may lead to a chattering effect. To eliminate the effect of chattering, a boundary layer control law may be used in place of the discontinuous control $\tilde v$ thus obtaining only the ultimate boundedness of the tracking error e.

4. Numerical examples

In this section, we illustrate the performance of the proposed controller (25)–(26) using the data of the Experimental Direct Drive Arm (EDDA manipulator) ($n = 2$) from the Institute of Robotics and Informatics of the Braunschweig University, Germany. In the simulations, SI units are used. The components of dynamic equations of this manipulator are as follows $M = \begin{bmatrix}\theta_1 + 0.6\theta_4\cos(q_2) & \theta_3 + 0.3\theta_4\cos(q_2)\\ \theta_3 + 0.3\theta_4\cos(q_2) & \theta_3\end{bmatrix}$; $B = \begin{bmatrix}-1\\ 0\end{bmatrix}0.3\theta_4\sin(q_2)$; $C = \begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}0.3\theta_4\sin(q_2)$; $G = g\begin{pmatrix}\theta_2\cos(q_1) + \theta_4\cos(q_1+q_2)\\ \theta_4\cos(q_1+q_2)\end{pmatrix}$; g stands for the gravitational acceleration; parameters $\theta_i, i = 1:4$ take the following nominal values: $\theta_1 = 3.1, \theta_2 = 9.5, \theta_3 = 0.24, \theta_4 = 0.77$. Our estimates are chosen as $\hat M = \frac{2}{\lambda_{\min}(M^{-1})+\lambda_{\max}(M^{-1})}\mathbb{I}_2$; $\hat F = 0$; $\lambda_{\min}(M^{-1}) = 0.27$; $\lambda_{\max}(M^{-1}) = 6$. The initial configuration and velocity of the manipulator are equal to $q(0) = (-\pi/4,\ \pi/4)^T, \dot q(0) = (0,\ 0)^T$, respectively. In order to speed up the convergence process of differentiator (25), we have chosen good initial guesses $y_1(0), y_2(0)$ in the numerical examples (which imply relation $T' \simeq 0$) based on the nominal values of our perturbed dynamic model. Consequently, differentiator (25) was run with the following initial values: $y_1(0) = \dot q(0), y_2(0) = \ddot q(0) = (-22.1,\ 5.7)^T, v(0) = (0,\ 0)^T$. In the first experiment, the (unbounded) disturbance term D takes the form $D = -2(q-q(0)) - 2\dot q$ with $\dot D = -2\dot q - 2\ddot q$. Consequently, α_0, α_1 may be estimated as $\alpha_0 = 2\|q - q(0)\| + 2\|y_1\|, \alpha_1 = 2\|y_1\| + 2\|y_2\|$. In order to simplify numerical computations, rough conservative estimations of $c_i, i = 1:5$ have been assumed. Hence, positive constant coefficients $c_i, i = 1:5$ were chosen as follows $c_1 = 10, c_2 = 150, c_3 = 25, c_4 = 5$ and $c_5 = 10$, respectively. The task is to make the manipulator follow the desired trajectory $q_d(t) = (\cos(t),\ \sin(t))^T$. The controller gains $\lambda_0, \lambda_1, \lambda_2, \Lambda, \kappa, \beta$ equal $\lambda_0 = 1, \lambda_1 = 21, \lambda_2 = 16, \Lambda = 13, \kappa = 6, \beta = \frac{5}{7}$, respectively.

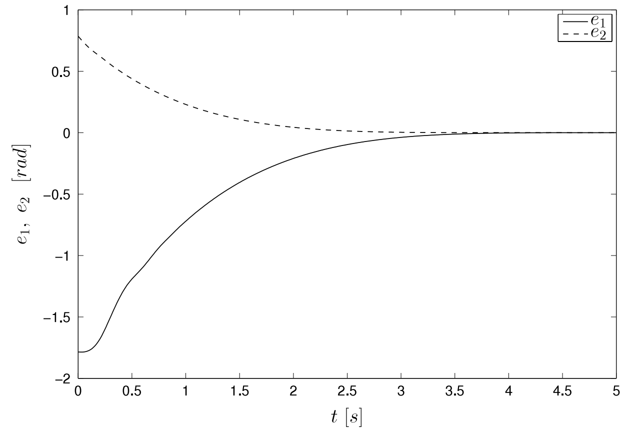

Fig. 1. Position errors e for controller (25)–(26).

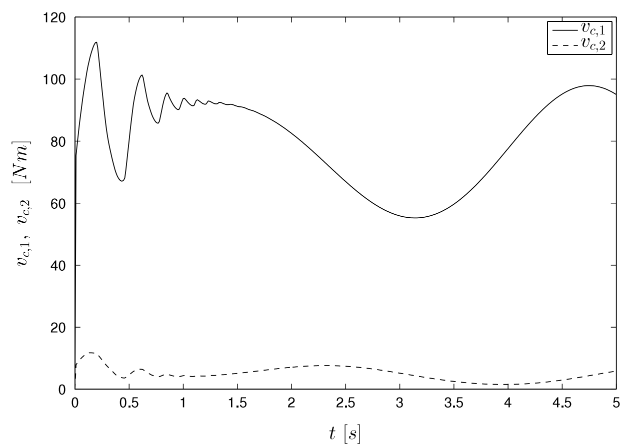

Fig. 2. Control vector v_c for controller (25)–(26).

The results of computer simulations are presented in Figs. 1–2. As is seen from Fig. 1, controller (25)–(26) stabilises the equilibrium $e = 0$ in finite time. Continuous and chattering-free controls v_c are shown in Fig. 2. In the second experiment, the Coulomb friction term $5\,\mathrm{sign}(\dot{q})$ (the same term was analysed in Jin et al., 2009) has been added to disturbed manipulator dynamic equations considered in the previous experiment. The same controller (25)–(26) as that from the first experiment with the same controller gains has been applied in the second experiment. For better visualisation of the accuracy of the tracking errors, the transient phase of approaching the manipulator to desired trajectory q_d is omitted. The results of computer simulations are given in Figs. 3–4. As is seen from Figs. 3–4, the control performance does not degrade even at the time instances corresponding to discontinuity of the term $5\,\mathrm{sign}(\dot{q})$. The reason is that control law (25)–(26) becomes the continuous (but not smooth) filter with respect to v (continuous acceleration y_2 approximates possibly best in Chebyshev's sense the discontinuous disturbance). Note that $\dot{q}(t) = y_1(t), \ddot{q}(t) = y_2(t)$ for $t > T', y_2(t)$ depends on v in (25) and hence the right hand side of differential equation (17) is dependent on v, too. Although normal distribution $N(0,\ 1)$ does not provide locally bounded Lebesgue measurable noise, we have tested our controller under conditions of the first experiment and for $\eta_i(t) = 10^{-8}X(t); X(t) \sim N(0,\ 1), i = 1, 2$. Measurement noise η was added to joint position obtained from integration of state equations (7), (14). The result of simulation is given in Fig. 5 which indicates a good performance of controller (25)–(26) subject to measurement noise.

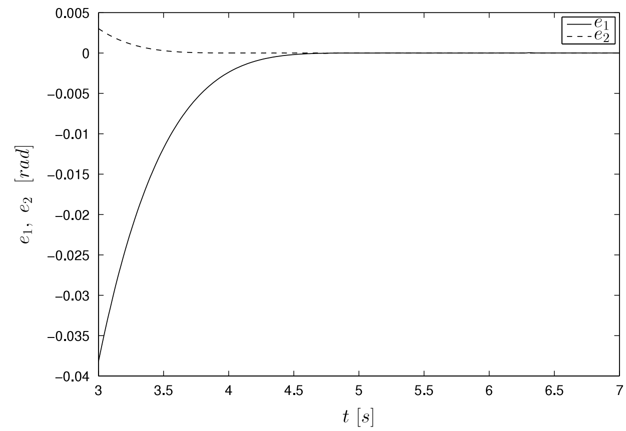

Fig. 3. Position errors e for controller (25)–(26) with the additional Coulomb friction term.

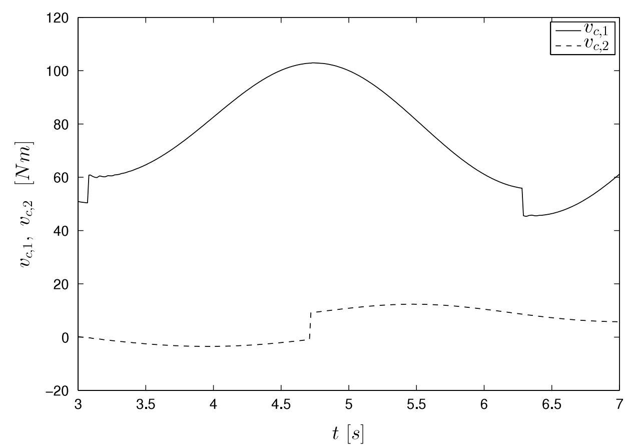

Fig. 4. Control vector v_c for controller (25)–(26) with the additional Coulomb friction term.

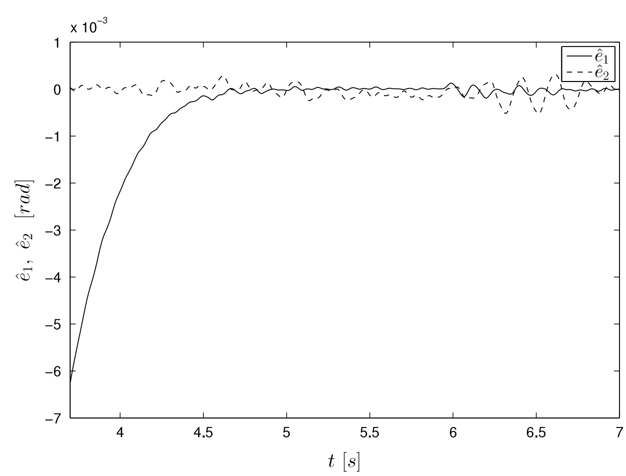

Fig. 5. Estimated position errors $\hat{e}$ for controller (25)–(26) with measurement noise.

5. Conclusions

A new class of absolutely continuous TSM controllers with the finite-time convergence property of the trajectory tracking by n-DoF rigid robotic manipulator has been proposed in this paper. Moreover, a novel TSM manifold, making it possible to simultaneously apply both the first- and second-order sliding mode control techniques with their advantages, was incorporated into the control scheme. Although our dynamically computed torque technique needs knowledge about the system equations of the robot, the approach is able to handle uncertainty (in dynamics and disturbance) occurring in the system. It is worth to emphasise the fact that the controllers proposed herein are able to cope with globally unbounded disturbances acting on the robotic manipulators. One of the challenging works for further research is to design a class of robust controllers with the finite-time convergence property which track the manipulator trajectories given in a task (and not joint) space. If this is the case, the control matrix may not be positive definite.

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